Introduction to Geometric Langlands

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Contents

1 Introduction: 02/09/2025 (Minghan)

Scribe: Minghan Sun

Prerequisites

There are three main prerequisites for the course:

1. Algebraic geometry on the level of Hartshorne.
2. Algebraic topology.
3. Representation theory.

1.1 Abelian Extensions of $\mathbb{Q}$

In his proof of Quadratic Reciprocity, Gauss uses the following result.

Theorem 1 (Gauss sums)**.** *Suppose p is a prime. We have three identities.*

1. *If $p \equiv 1 \pmod 4$, then*
$$\pm\sqrt{p} = \sum_{x\in(\mathbb{Z}/p)^\times} \left(\frac{x}{p}\right)\zeta_p^x. \tag{1}$$
Here $\zeta_p = e^{2\pi i/p}$ is a primitive p-th root of unity and $\left(\frac{x}{p}\right)$ is the Legendre symbol, i.e.
$$\left(\frac{x}{p}\right) = \begin{cases} 1 & \text{if } x \in \mathbb{F}_p^2, \\ -1 & \text{else.} \end{cases} \tag{2}$$
2. *If $p \equiv 3 \pmod 4$, then*
$$\pm i\sqrt{p} = \sum_{x\in(\mathbb{Z}/p)^\times} \left(\frac{x}{p}\right)\zeta_p^x. \tag{3}$$
3. *If $p = 2$, then*
$$\pm\sqrt{2} = \zeta_8 + \zeta_8^7. \tag{4}$$

Example 1. *Suppose $p = 3$. Then we have*
$$\begin{aligned} \sum_{x\in(\mathbb{Z}/3)^\times} \left(\frac{x}{3}\right)\zeta_3^x &= \left(\frac{1}{3}\right)\zeta_3 + \left(\frac{2}{3}\right)\zeta_3^2 \\ &= \zeta_3 - \zeta_3^2 \\ &= \sqrt{3}i, \end{aligned} \tag{5}$$
as asserted in Theorem 1.

Remark 1. *The interested reader can find proofs of the Gauss sums (Theorem 1) in Serre's A Course in Arithmetic.*

Theorem 1 implies the following general fact.

Proposition 1. *Suppose p is a prime. Then*
$$\mathbb{Q}(\sqrt{p}) \subset \begin{cases} \mathbb{Q}(\zeta_p) & \text{if } p \equiv 1 \pmod 4, \\ \mathbb{Q}(\zeta_{4p}) & \text{if } p \equiv 3 \pmod 4, \\ \mathbb{Q}(\zeta_8) & \text{if } p = 2. \end{cases} \tag{6}$$

Let's say a few words about the structure of the cyclotomic fields $\mathbb{Q}(\zeta_N)$.

Theorem 2 (structure of cyclotomic fields)**.** *For all N, $\mathbb{Q}(\zeta_N)$ itself is an abelian extension of $\mathbb{Q}$. Moreover, the map $\chi : \mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{Q}) \to (\mathbb{Z}/n)^\times$ given by $\sigma \mapsto \chi(\sigma)$ such that $\sigma(\zeta_N) = \zeta_N^{\chi(\sigma)}$ is an isomophism.*

One of the greatest gems of classical number theory is the following generalization of Proposition 1.

Theorem 3 (Kronecker-Weber)**.** *Suppose $F/\mathbb{Q}$ is an abelian Galois extension. Then there exists some N such that $F \subset \mathbb{Q}(\zeta_N)$.*

Remark 2. *So far, we have given a complete and explicit description of all abelian Galois extensions of $\mathbb{Q}$. In the next subsection, we will begin to discuss class field theory (CFT), which concerns abelian Galois extensions of what are called "global" and "local" fields.*

1.2 Basic CFT

We begin by defining global and local fields.

Definition 1 (number fields and function fields)**.** *A* number field *is a finite extension of $\mathbb{Q}$. A* function field *is a field of the form $\mathbb{F}_q(X_0)$, where X_0 is a geometrically connected smooth projective curve and $\mathbb{F}_q(X_0)$ is the field of rational functions on X_0.*

Definition 2 (global fields)**.** *A* global field *is either a number field or a function field.*

Definition 3 (local fields)**.** *A* local field *is $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{Q}_p$, or a finite extension of $\mathbb{F}_q((t))$.*

Remark 3. *We can equivalently define a local field as a locally compact (topological) field.*

Definition 4 (places of global fields)**.** *Suppose F is a global field. A* place *of F is a norm ν on F such that F_ν (the completion of F with respect to ν) is a local field.*

Example 2. *$\mathbb{Q}$ has a place whose completion is $\mathbb{R}$ and also a place for each prime p whose completion is $\mathbb{Q}_p$.*

The places of $\mathbb{F}_q(X_0)$ are indexed by closed points on the curve X_0.

Definition 5 (adeles)**.** *Suppose F is a global field and P the set of places of F. We define the* adeles *of F, denoted $\mathbb{A}_F$, as the additive subgroup of the product*

$$\prod_{\nu \in P} F_\nu \tag{7}$$

consisting of elements $x = (x_\nu)$ such that $x_\nu \in \mathcal{O}_\nu$ almost everywhere (i.e. for all but finitely many places).

Remark 4. *In Definition 5, if ν is a nonarchimedean place (i.e. not $\mathbb{R}$ or $\mathbb{C}$), then $\mathcal{O}_\nu$ denotes the ring of integers of F_ν.*

Remark 5. *Note that we have a natural diagonal embedding $\iota : F \to \mathbb{A}_F$ mapping F to each of its completions.*

When we study global fields in number theory, we often first work with local fields and collect our results into the adeles. Then, the real work consists in understanding the map ι.

Lemma 1 (adeles of $\mathbb{Q}$)**.** *We have*

$$\mathbb{A}_\mathbb{Q} = (\hat{\mathbb{Z}} \otimes \mathbb{Q}) \times \mathbb{R} = (\hat{\mathbb{Z}} \times \mathbb{R}) \otimes \mathbb{Q}, \tag{8}$$

where $\hat{\mathbb{Z}} = \varprojlim_{n \to \infty} \mathbb{Z}/n$.

Proof. Left as an exercise to the reader. □

Proposition 2. *Suppose F is a global field. We have two facts.*

1. *F, regarded as a subspace of $\mathbb{A}_F$ via the diagonal embedding $\iota : F \to \mathbb{A}_F$, is discrete.*
2. *$\mathbb{A}_F/F$ is compact.*

Remark 6. *Proposition 2 roughly says that the relationship of F to $\mathbb{A}_F$ is somewhat analogous to the relationship of $\mathbb{Z}$ to $\mathbb{R}$.*

2 More Adèles, CFT, etc.: 04/09/2025

Scribe: Max Steinberg

Today, we will discuss more on adèles, CFT (not conformal field theory), etc.

2.1 Structure of Non-Archimedean Local Fields

Let K be such a field. By definition (or by classification), K is a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_\mathrm{q}((t))$.

2.1.1 General structure

We have $\mathcal{O}_K \subset K$ the ring of integers in K. It is the unique compact open, integrally closed subring.

Example 3. *$\mathcal{O}_{\mathbb{Q}_p} = \mathbb{Z}_p$, $K/\mathbb{Q}_p$ is finite, $\mathcal{O}_K$ is the integral closure of $\mathbb{Z}_p$. If $K = \mathbb{F}_\mathrm{q}((t))$, $\mathcal{O}_K = \mathbb{F}_\mathrm{q}[\![t]\!]$.*

We can think of $\mathcal{O}_K = \left\{x \in K \middle| |x| \leq 1\right\}$ w.r.t a suitable norm (i.e. the norm generating the topology). We recall that $\mathcal{O}_K$ is a DVR (discrete valuation ring), i.e. $\exists! \mathfrak{m} \neq 0$ prime (maximal) such that $\mathfrak{m} = (\varpi)$ (called the "uniformiser") and $\mathcal{O}_K/\mathfrak{m}_K = k_K$, the residue field, which is a finite field.

If K is a local field, a finite extension L/K (which automatically makes L a local field) is **unramified** if $\mathcal{O}_L/\mathcal{O}_K$ is an <u>étale extension</u> ($\iff$ ϖ_K generates $\mathfrak{m}_{\mathcal{O}_L}$).

Reminder: étale extensions are a generalisation of separable field extensions to commutative rings.

Example 4. *$\mathbb{F}_{\mathrm{q}^\mathrm{n}}((t))$ is unramified over $\mathbb{F}_\mathrm{q}((t))$. All unramified extensions have this form.*

It is a fact that the following groupoids are isomorphic:

$$\begin{aligned} \{L/K\,\text{unramified}\} &\xrightarrow{\simeq} \{k'/k_K\} \\ L &\mapsto \mathcal{O}_L/\mathfrak{m}_L \end{aligned}$$

Motto: unramified extensions of K are in bijection with extensions of its residue field.

Example 5. $(\mathbb{Z}_3)_3 \simeq \mathbb{F}_3$ *(where the left-hand side denotes $\mathbb{Z}_3$, the 3-adic integers, localised at the prime 3), $\mathbb{F}_9\,/\,\mathbb{F}_3$, $\mathbb{F}_9 = \mathbb{F}_3[\sqrt{2}]$ and the corresponding unramified extension is $\mathbb{Q}_3[\sqrt{2}]$.*

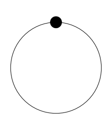
Reminder: if k is a field, $\mathrm{Gal}(k) := \mathrm{Gal}(k^{sep}/k) = \varprojlim \mathrm{Gal}(k'/k)$ with k'/k a finite Galois extension with $k^{sep}/k'/k$. If k is a finite field, $k = \mathbb{F}_\mathrm{q}, q = |k|$, there is a distinguished element $\mathrm{Fr} = \mathrm{Fr}_q \in \mathrm{Gal}(k)$, with $\mathrm{Fr}(x) = x^q, x \in \overline{k}$. This fixes $\mathbb{F}_\mathrm{q}$ because $x^q = x$ exactly for $x \in \mathbb{F}_\mathrm{q}$, and is an automorphism of $\mathbb{F}_\mathrm{q}$ by finiteness and $(x+y)^q = x^q + y^q$. There is a canonical map

$$\begin{aligned} \mathbb{Z} &\to \mathrm{Gal}(\mathbb{F}_\mathrm{q}) \\ 1 &\mapsto \mathrm{Fr}_q \end{aligned}$$

The profinite completion of this map is an isomorphism $\hat{\mathbb{Z}} \simeq \mathrm{Gal}(\mathbb{F}_\mathrm{q})$.

Remark: the **Weil group** $W_{\mathbb{F}_\mathrm{q}}$ of $\mathbb{F}_\mathrm{q}$ is $\mathbb{Z} \subset \mathrm{Gal}(\mathbb{F}_\mathrm{q})$ generatd by the Frobenius. It is an error in nature that $\mathrm{Gal}(\mathbb{F}_\mathrm{q}) = \hat{\mathbb{Z}}$ not $\mathbb{Z}$ and the Weil group corrects this in an ad-hoc way.

Remark: You can picture $\operatorname{Spec} \mathbb{F}_\mathrm{q}$ as



with the idea that $\pi_1 = \mathbb{Z}$ just like how $\mathrm{Gal}(\mathbb{F}_\mathrm{q})$ "=" $\mathbb{Z}$.

Claim. Let K be non-Archimedean, k its residue field. Then the earlier discussion gives a map $\mathrm{Gal}(K) \to \mathrm{Gal}(k) \simeq \hat{\mathbb{Z}}$: We construct $K^{unr} \subseteq K^{sep}$ the union of all finite unramified extensions. Then $\mathrm{Gal}(K) := \mathrm{Aut}(K^{sep}/K) \to \mathrm{Aut}(K^{unr}/K) \simeq \mathrm{Aut}(\overline{k}/k)$ by the previous discussion.

Definition 6 (Weil group)**.** *For K a local field, the **Weil group** of K, W_K, is the preimage of $\mathbb{Z} \subset \hat{\mathbb{Z}}$ under this map, topologised in a natural way.*

2.2 Main Theorem of Local CFT

Theorem 4 (Main Theorem of Local CFT)**.** $W_K^{ab} \simeq K^\times$ *"canonically". In fact, the following diagramme commutes:*

$$\begin{tikzcd} W_K^{ab} \arrow[rr, "\simeq"] \arrow[dr] & & K^\times \arrow[dl, "v"] \\ & W_k = \mathbb{Z} & \end{tikzcd}$$

Remark: (by inspection) this also holds for Archimedean local fields.

A note on the structure of $K^\times$: there is a valuation $K^\times \xrightarrow{v} \mathbb{Z}$, the unique valuation with $v(\varpi) = 1$. In fact, $\ker v = \mathcal{O}_K^\times$.

Corollary 1. $\mathrm{Gal}(K)^{ab} \simeq (K^\times)^\wedge$*, where* $(-)^\wedge$ *is the profinite completion functor.*

2.3 Global Setting

Let F be a global field and v a place of F, and F_v the corresponding local field.

$$\begin{tikzcd} F \arrow[r, hook] \arrow[d, hook] & F_v \arrow[d, "\subseteq"] \\ F^{sep} \arrow[r, "\subseteq"] & F_v^{sep} \end{tikzcd}$$

This gives rise to a map $\mathrm{Gal}(F_v) \to \mathrm{Gal}(F)$ (well-defined up to conjugation). (If you want to learn more, look up "decomposition group.")

If we want to understand $\mathrm{Gal}(F)^{ab}$, we know it receives a map $W_{F_v}^{ab} \to \mathrm{Gal}(F_v)^{ab} \to \mathrm{Gal}(F)^{ab}$. We can write the diagramme:

$$\begin{tikzcd} F_v^\times \arrow[r] \arrow[dr] & W_{F_v}^{ab} \arrow[d] \\ & \mathrm{Gal}(F_v)^{ab} \arrow[d] \\ & \mathrm{Gal}(F)^{ab} \end{tikzcd}$$

These combine into a map $\mathbb{A}_F^\times \to \mathrm{Gal}(F)^{ab}$.

Theorem 5 (Main Theorem of Global CFT)**.** *We have the following:*

1. *The composition* $F^\times \subset \mathbb{A}_F^\times \to \mathrm{Gal}(F)^{ab}$ *is trivial ("Artin reciprocity")*

2. *The induced map* $(\mathbb{A}_F^\times / F^\times)^\wedge \simeq \mathrm{Gal}(F)^{ab}$.

Example 6. $F = \mathbb{Q}$*,* $\mathbb{A}_\mathbb{Q}^\times = \mathbb{R}^\times \times \prod' \mathbb{Q}_p^\times$*, and* $\mathbb{A}_\mathbb{Q}^\times / \mathbb{Q}^\times = \mathbb{R}^{\geq 0} \times \prod \mathbb{Z}_p^\times$*. The map is* $(x_\infty, (x_p)) \mapsto \left(|x_\infty| \prod p^{-v_p(x_p)}, \left(|x_p|_p \prod p^{-v_p(x_p)}\right)\right)$.

Then $(\mathbb{A}_\mathbb{Q}^\times / \mathbb{Q}^\times)^\wedge \simeq \prod \mathbb{Z}_p^\times = \hat{\mathbb{Z}}^\times$*. Last class we discussed* $\mathrm{Gal}(\mathbb{Q}(\zeta_n)/\mathbb{Q}) = (\mathbb{Z}/n)^\times$*, and passing to the inverse limit gives rise to Global CFT.*

3 Étale Fundamental Groups and the Analytic Jacobian: 09/09/2025

Scribe: Zachary Carlini

3.1 Étale Fundamental Groups

For a more detailed treatment of this material, see [Gro71].

Let X be a connected scheme (which is usually assumed to be normal). Let K be a separably closed field, and let x_0 be a K-point of X. From this data, the theory of ètale fundamental groups produces a profinite group $\pi_1^{\text{ét}}(X, x_0)$. We will suppress x_0 from the notation when the choice of x_0 is unimportant, so we will write $\pi_1^{\text{ét}}(X)$ instead.

Example 7. *For k a field and K its separable closure,*

$$\pi_1^{ét}(\operatorname{Spec} k, \operatorname{Spec} K) \cong \operatorname{Gal}(K/k).$$

Example 8. *For X a smooth variety over $\mathbb{C}$ and $x_0 \in X$ a closed point,*

$$\pi_1^{ét}(X, x_0) \cong \pi_1^{top}(X^{an}, x_0)^\wedge,$$

where X^{an} denotes the analytification of X, π_1^{top} denotes the topological fundamental group, and $^\wedge$ denotes the profinite completion.

The étale fundamental group has the following characterization: an action of the étale fundamental group on any finite set S corresponds to a finite étale cover $Y \to X$ together with a bijection between S and the set of sections of the fiber $Y \times_X x_0 \to x_0$. Equivalently, we can define $\pi_1^{\text{ét}}(X, x_0)$ to be the automorphism group of the functor $\mathrm{Sch}_{/X}^{\text{finite,étale}} \to \mathrm{FinSet}$ which sends a cover $\pi : Y \to X$ to the set of $y \in Y(K)$ with $\pi(y) = x_0$.

If $\tilde{X} \to X$ is a finite étale cover between connected, smooth varieties over $\mathbb{C}$, then the induced map $\tilde{X}^{\text{an}} \to X^{\text{an}}$ is a finite covering in the sense of algebraic topology. The Riemann Existence Theorem provides a converse.

Theorem 6 (Riemann Existence)**.** *If X is a smooth variety over $\mathbb{C}$, and $\tilde{X}^{an} \to X^{an}$ is a finite cover, then $\tilde{X}^{an}$ admits a unique variety structure $\tilde{X}$ compatible with the variety structure on X (i.e. regular functions on open subsets of X pull back to regular functions on $\tilde{X}$), and the projection $\tilde{X} \to X$ is a finite étale cover.*

Example 8 is a corollary of the Riemann Existence Theorem.

3.2 Geometric CFT

Now, suppose K is a non-Archimedean local field with ring-of-integers $\mathcal{O}_K$. Let k_K be the residue field of K, i.e. the quotient of $\mathcal{O}_K$ by its maximal ideal. Recall from yesterday that we had a bijection:

$$\begin{aligned} \{L/K \text{ unramified}\} &\xrightarrow{\simeq} \{k'/k_K\} \\ L &\mapsto \mathcal{O}_L/\mathfrak{m}_L \end{aligned}$$

We may now reformulate this as a statement about fundamental groups. Namely, the map $\operatorname{Spec} \mathcal{O}_K \to \operatorname{Spec} k_K$ induces an isomorphism $\pi_1^{\text{ét}}(\operatorname{Spec} \mathcal{O}_K) \xrightarrow{\sim} \pi_1^{\text{ét}}(\operatorname{Spec} k_K)$.

We also have a map $\operatorname{Spec} K \to \operatorname{Spec} \mathcal{O}_K$. This induces a map $\pi_1^{\text{ét}}(\operatorname{Spec} K) \to \pi_1^{\text{ét}}(\operatorname{Spec} \mathcal{O}_K)$, which we can identify with the surjection $\operatorname{Gal}(K) \twoheadrightarrow \operatorname{Gal}(k)$ mentioned yesterday. This gives a geometric interpretation of our discussion of local class field theory.

Next, we will give a geometric interpretation of global class field theory in the case of function fields. Fix a finite field $\mathbb{F}_q$ and a smooth projective curve X_0 over $\mathbb{F}_q$. Let F be the field of rational functions on X_0, and let $\mathbb{A}$ be its ring of adèles. Recall from yesterday that we had an isomorphism

$$\operatorname{Gal}(F)^{\text{ab}} \cong \left(\mathbb{A}^\times / F^\times\right)^\wedge. \tag{9}$$

Fix an algebraic closure $\bar{\mathbb{F}}_q$ of $\mathbb{F}_q$, and let X denote the base-change $X = X_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Let $U_0 \subset X_0$ be open, and let $U = U_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. We define the Weil group W_{U_0} of U_0 to be the preimage of $\mathbb{Z} \subset \hat{\mathbb{Z}}$ along the map $\pi_1^{\text{ét}}(U_0) \to \pi_1^{\text{ét}}(\operatorname{Spec} \mathbb{F}_q)$ induced by $U_0 \to \text{pt}$. There is an exact sequence:

$$1 \to \pi_1^{\text{ét}}(U) \to W_{U_0} \to \mathbb{Z} \to 0,$$

and we endow W_{U_0} with the topology that makes it a topological disjoint union of cosets of $\pi_1^{\text{ét}}(U)$ with its profinite topology (note that this is not the same as the subspace topology coming from $W_{U_0} \subset \pi_1^{\text{ét}}(U_0)$).

For every closed point $x \in X_0$, let $\mathcal{O}_x$ be the ring of integers in the completion of F at x. There is a homomorphism $\mathcal{O}_x^\times \to \mathbb{A}^\times$ which sends $f \in \mathcal{O}_x^\times$ to the adèle which is f at place x and 1 everywhere else. Global class field theory produces an isomorphism:

$$W_{U_0}^{\text{ab}} \cong F^\times \backslash \mathbb{A}^\times / \prod_{x \in U_0} \mathcal{O}_x^\times. \tag{10}$$

The isomorphism (9) is obtained from (10) by taking profinite completions and taking the limit as $U_0 \to \emptyset$.

Question 1 (Deep)**.** *What is the correct analog of W_{U_0} for a number field? This is referred to as the "Langlands group" in the literature. Constructing the Langlands group and proving that it has the desired properties is still an open problem.*

For ease of exposition, we will now specialize to the case where $U_0 = X_0$. We want:

$$W_{X_0}^{\text{ab}} \cong F^\times \backslash \mathbb{A}^\times / \prod_{x \in X_0} \mathcal{O}_x^\times,$$

so we should unwind $F^\times \backslash \mathbb{A}^\times / \prod_{x \in X} \mathcal{O}_x^\times$.

First, for each $x \in X_0$, let K_x be the completion of F at x. Then $\mathcal{O}_x$ is a DVR with fraction field K_x, so $K_x^\times / \mathcal{O}_x^\times \cong \mathbb{Z}$. Since $\mathbb{A}$ is, by definition, the restricted direct product of the K_x, we have:

$$\mathbb{A}^\times / \prod_{x \in X_0} \mathcal{O}_x^\times \cong \bigoplus_{x \in X_0} \mathbb{Z} = \operatorname{Div}(X_0).$$

Next, the map $F^\times \to K_x^\times / \mathcal{O}_x^\times \cong \mathbb{Z}$ sends a rational function f to its order of vanishing at x, so $F^\times \to \mathbb{A}^\times / \prod_{x \in X} \mathcal{O}_x^\times \cong \operatorname{Div}(X)$ sends a rational function to its divisor. Therefore, we have:

$$F^\times \backslash \mathbb{A}^\times / \prod_{x \in X_0} \mathcal{O}_x^\times \cong F^\times \backslash \operatorname{Div}(X_0) \cong \operatorname{Pic}(X_0),$$

where $\operatorname{Pic}(X_0)$ denotes the group of algebraic vector bundles on X_0 with the group operation of $\otimes$. For a reference on divisors and line bundles, see [Har77][Chapter II.6].

Now, unramified global class field theory can be stated as an isomorphism:

$$W_{X_0}^{\text{ab}} \cong \operatorname{Pic}(X_0).$$

Moreover, the map $W_{X_0} \to \mathbb{Z}$ identifies under this isomorphism with the map that sends a line bundle to its degree. We will construct the map $W_{X_0} \to \operatorname{Pic}(X_0)$, but first we will try to understand an easier toy model.

3.3 A Toy Model for Geometric Class Field Theory

Let X be a smooth, projective curve over $\mathbb{C}$. From [Ser56], we have an isomorphism $\operatorname{Pic}(X) \xrightarrow{\sim} \operatorname{Pic}(X^{\text{an}})$, and from general theory we have $\operatorname{Pic}(X^{\text{an}}) \cong H^1(X, \mathcal{O}_X^\times)$.

From complex analysis, there is an exact sequence of sheaves:

$$0 \to \underline{\mathbb{Z}(1)}_{X^{\text{an}}} \to \mathcal{O}_{X^{\text{an}}} \xrightarrow{\exp} \mathcal{O}_{X^{\text{an}}}^\times \to 0,$$

where $\mathbb{Z}(1) = 2\pi i \mathbb{Z} \subset \mathbb{C}$. Taking global sections gives a long exact sequence:

$$\begin{aligned} 0 &\to H^1(X^{\text{an}}, \mathbb{Z}(1)) \to H^1(X^{\text{an}}, \mathcal{O}) \\ &\to H^1\left(X^{\text{an}}, \mathcal{O}^\times\right) \xrightarrow{d} H^2(X^{\text{an}}, \mathbb{Z}(1)) \to 0 \end{aligned} \tag{11}$$

(Here, we have used the fact that the exponential map induces a surjection from $H_0(X, \mathcal{O}) \cong \mathbb{C}$ to $H_0(X, \mathcal{O}^\times) \cong \mathbb{C}^\times$). We may identify $H^1(X^{\mathrm{an}}, \mathcal{O}^\times) \cong \mathrm{Pic}(X^{\mathrm{an}})$ and $H^2(X^{\mathrm{an}}, \mathbb{Z}(1)) \cong \mathbb{Z}$, in which case d becomes $\mathrm{Pic}(X^{\mathrm{an}}) \xrightarrow{\deg} \mathbb{Z}$. Thus, $\ker(d)$ is the group of holomorphic line bundles on X^{an} with degree 0, which we denote $\mathrm{Jac}(X^{\mathrm{an}})$. From (11), we obtain a short exact sequence:

$$0 \to H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \to H^1(X^{\mathrm{an}}, \mathcal{O}) \to \mathrm{Jac}(X^{\mathrm{an}}) \to 0.$$

If g is the genus of X, then $H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \cong \mathbb{Z}^{2g}$ and $H^1(X^{\mathrm{an}}, \mathcal{O}) \cong \mathbb{C}^g$, so $\mathrm{Jac}(X^{\mathrm{an}}) \cong \mathbb{C}^g / \mathbb{Z}^{2g}$ is the quotient of a complex vector space by a lattice. In particular, $\mathrm{Jac}(X^{\mathrm{an}})$ acquires a natural complex-analytic geometry, and with respect to this geometry, $\mathrm{Jac}(X^{\mathrm{an}})$ is a compact, complex Lie-group. Over $\mathbb{R}$, $\mathrm{Jac}(X^{\mathrm{an}})$ is isomorphic as a Lie group to $(S^1)^{\times 2g}$, but the complex-analytic structure of $\mathrm{Jac}(X)$ will depend on the complex-analytic structure of X^{an}.

4 Geometric Class Field Theory: 11/09/2025

Scribe: Joakim Færgeman

4.1 Geometric Class Field Theory In Terms Of Fundamental Groups

Let us take a moment to orient ourselves. For X_0 a smooth projective curve over a finite field $\mathbb{F}_q$, we saw that class field theory for the global field $\mathbb{F}_q(X_0)$ took the following form.

We defined the Weil group W_{X_0} that sits in a short exact sequence:

$$1 \to \pi_1^{\text{ét}}(X) \to W_{X_0} \to \mathbb{Z} \to 0.$$

Here, $X/\overline{\mathbb{F}_q}$ denotes the base-change of X_0 to $\overline{\mathbb{F}_q}$. Class field theory says that we have an isomorphism of groups:

$$W_{X_0}^{\mathrm{ab}} \simeq \mathrm{Pic}(X_0),$$

where we remind that $\mathrm{Pic}(X_0)$ denotes the group of line bundles on X_0 up to isomorphism.

We want a version of class field theory that generalizes to the complex numbers (and in fact any field). We will refer to this generalization as geometric class field theory (GCFT). Let $X/\mathbb{C}$ be a smooth projective curve over $\mathbb{C}$. We have:

$$\mathrm{Jac}(X)^{\mathrm{an}} = \mathrm{Ker}\big(H^1(X^{\mathrm{an}}, \mathcal{O}^*) \to H^2(X^{\mathrm{an}}, \mathbb{Z}(1))\big).$$

This is the group of degree 0 line bundles on X.[1] We expressed

$$\mathrm{Jac}(X)^{\mathrm{an}} \simeq H^1(X^{\mathrm{an}}, \mathcal{O})/H^1(X^{\mathrm{an}}, \mathbb{Z}(1))$$

as a complex manifold. We remind that if $g = g(X)$ denotes the genus of X, then $\dim_{\mathbb{C}} H^1(X^{\mathrm{an}}, \mathcal{O}) = \dim_{\mathbb{C}} H^0(X^{\mathrm{an}}, \Omega^1)$. Fix a point $x_0 \in X$. This induces an Abel-Jacobi map:

$$\mathrm{AJ}_{x_0} : X^{\mathrm{an}} \to \mathrm{Jac}(X)^{\mathrm{an}}, \;\; x \mapsto \mathcal{O}(x - x_0).$$

One version of geometric class field theory states:

Theorem 7 (One version of GCFT)**.** *The map AJ_{x_0} induces an isomorphism of groups $\pi_1^{\mathrm{top}}(X^{\mathrm{an}})^{\mathrm{ab}} \xrightarrow{\simeq} \pi_1^{\mathrm{top}}(\mathrm{Jac}(X)^{\mathrm{an}})$.*

The rest of this subsection is devoted to sketching the proof of this theorem. We will see that the proof comes down to spelling out a certain compatibility between Poincaré duality and Serre duality for X^{an}.

Recall that on general grounds, we have a canonoical isomorphism of abelian groups:

$$\pi_1^{\mathrm{top}}(X^{\mathrm{an}})^{\mathrm{ab}} \xrightarrow{\simeq} H_1(X^{\mathrm{an}}, \mathbb{Z}).$$

As we saw above, the universal cover of $\mathrm{Jac}(X)^{\mathrm{an}}$ is $H^1(X^{\mathrm{an}}, \mathcal{O}) \simeq \mathbb{C}^g$. Thus, we have an isomorphism:

$$\pi_1^{\mathrm{top}}(\mathrm{Jac}(X)^{\mathrm{an}}) \simeq H^1(X^{\mathrm{an}}, \mathbb{Z}(1)).$$

Moreover, Poincaré duality states that the cup product map gives a non-degenerate pairing:

$$H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \otimes H^1(X^{\mathrm{an}}, \mathbb{Z}) \to H^2(X^{\mathrm{an}}, \mathbb{Z}(1)).$$

Hence, we have $H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \simeq \mathrm{Hom}_{\mathbb{Z}}(H^1(X^{\mathrm{an}}, \mathbb{Z}), \mathbb{Z}) \simeq H_1(X^{\mathrm{an}}, \mathbb{Z})$.[2] Combining the above, we see that there is an abstract isomorphism:

$$\pi_1^{\mathrm{top}}(X^{\mathrm{an}})^{\mathrm{ab}} \simeq \pi_1^{\mathrm{top}}(\mathrm{Jac}(X)^{\mathrm{an}}).$$

[1] Or equivalently, the complex line bundles that are topologically trivial.

[2] The last isomorphism follows from the universal coefficient theorem using that $H_0(X^{\mathrm{an}}, \mathbb{Z}) \simeq \mathbb{Z}$ is torsion-free.

The "real" content of geometric class field theory is that this isomorphism is induced by the Abel-Jacobi map AJ_{x_0}. We formulate this assertion as the combination of two claims, which we leave as exercises for the reader.

Serre Duality gives an isomorphism $H^1(X^{\mathrm{an}}, \mathcal{O}) \simeq H^0(X^{\mathrm{an}}, \Omega^1)^*$. We saw that Poincaré duality gives an isomorphism $H^1(X^{\mathrm{an}}, \mathbb{Z}(1)) \simeq H_1(X^{\mathrm{an}}, \mathbb{Z})$. Thus, we get:

$$\mathrm{Jac}(X)^{\mathrm{an}} \simeq H^0(X^{\mathrm{an}}, \Omega^1)^*/H_1(X^{\mathrm{an}}, \mathbb{Z}).$$

In particular, the combination of the two dualities provides an action

$$H_1(X^{\mathrm{an}}, \mathbb{Z}) \curvearrowright H^0(X^{\mathrm{an}}, \Omega^1)^*. \tag{12}$$

We now spell out this action. Given $\omega \in H^0(X^{\mathrm{an}}, \Omega^1)$ and a closed loop γ based at x_0, we may integrate ω along γ to get the number

$$\int_\gamma \omega \in \mathbb{C}.$$

As such, we get a pairing:

$$\pi_1^{\mathrm{top}}(X^{\mathrm{an}}) \times H^0(X^{\mathrm{an}}, \Omega^1) \to \mathbb{C}.$$

This factors through the pairing:

$$H_1(X^{\mathrm{an}}, \mathbb{Z}) \otimes H^0(X^{\mathrm{an}}, \Omega^1) \to \mathbb{C}.$$

Hence we get a map

$$H_1(X^{\mathrm{an}}, \mathbb{Z}) \to H^0(X^{\mathrm{an}}, \Omega^1)^*. \tag{13}$$

Claim 1. The map (13) induces the action (12).

Next, consider the Abel-Jacobi map:

$$\mathrm{AJ}_{x_0} : X^{\mathrm{an}} \to \mathrm{Jac}(X)^{\mathrm{an}} \simeq H^0(X^{\mathrm{an}}, \Omega^1)^*/H_1(X^{\mathrm{an}}, \mathbb{Z}).$$

Denote by $\widetilde{X^{\mathrm{an}}} \to X^{\mathrm{an}}$ the universal cover of X^{an}. The usual construction of $\widetilde{X^{\mathrm{an}}}$ is as the space of continuous maps $\gamma : [0,1] \to X, \gamma(0) = x_0$ up to endpoint-fixing homotopy. The composition

$$\widetilde{X^{\mathrm{an}}} \to X^{\mathrm{an}} \to H^0(X^{\mathrm{an}}, \Omega^1)^*/H_1(X^{\mathrm{an}}, \mathbb{Z})$$

lifts to a map

$$\widetilde{X^{\mathrm{an}}} \to H^0(X^{\mathrm{an}}, \Omega^1)^*. \tag{14}$$

Claim 2. The map (14) sends $[\gamma]$ to the functional $(\omega \mapsto \int_\gamma \omega)$.

As such, the map $X^{\mathrm{an}} \to H^0(X^{\mathrm{an}}, \Omega^1)^*/H_1(X^{\mathrm{an}}, \mathbb{Z})$ is defined by, for $x \in X$, choosing a path γ from x_0 to x and getting the corresponding functional on $H^0(X^{\mathrm{an}}, \Omega^1)^*$, which is well-defined up to translating by $H_1(X^{\mathrm{an}}, \mathbb{Z})$ via the action (12).

Combining the two claims proves the above version of geometric class field theory.

4.2 The moduli stack of line bundles

Our next goal is to give a purely algebro-geometric version of GCFT that generalizes to an arbitrary field. As such, let k be a field, and let X/k a smooth projective curve. Consider the functor

$$\mathrm{Bun}_{\mathbb{G}_m} : \{\text{commutative } k-\text{algs}\} \to \{\text{groupoids}\}$$

sending a commutative k-algebra A to the groupoid of line bundles on $X \times \mathrm{Spec}(A)$.

Here is a variant. Fix a point $x_0 \in X(k)$ (assuming it exists). Consider the functor

$$\underline{\mathrm{Pic}} : \{\text{commutative } k-\text{algs}\} \to \mathrm{Sets}$$

sending a commutative k-algebra A to the set $(\mathcal{L}, \alpha)$ consisting of a line bundle $\mathcal{L}$ on $X \times \mathrm{Spec}(A)$ and α is a trivialization of $\mathcal{L}$ on $\{x_0\} \times \mathrm{Spec}(A) \subset X \times \mathrm{Spec}(A)$.

Claim The functor $\mathrm{Bun}_{\mathbb{G}_m}$ is representable by an algebraic stack, and $\underline{\mathrm{Pic}}$ is representable by a scheme.[3]

Let us describe the basic structure of $\mathrm{Bun}_{\mathbb{G}_m}$. It has a degree map

$$\deg : \mathrm{Bun}_{\mathbb{G}_m} \to \mathbb{Z} = \bigsqcup_{n \in \mathbb{Z}} \mathrm{Spec}(k).$$

Informally, this map is simply given by taking the degree of a line bundle. Formally, however, one argues as follows. Suppose we are given an element of $\mathrm{Bun}_{\mathbb{G}_m}(A)$, that is, a line bundle $\mathcal{L}$ on $X \times \mathrm{Spec}(A)$. We need to produce an object of $\mathbb{Z}(A)$, that is, a locally constant function $\mathrm{Spec}(A) \to \mathbb{Z}$. Denote by $p_2 : X \times \mathrm{Spec}(A) \to \mathrm{Spec}(A)$ the projection onto the second factor. Since X is projective of dimension 1, the complex $Rp_{2,*}(\mathcal{L})$ is quasi-isomorphic to a two-term complex $\mathcal{E}^0 \to \mathcal{E}^1$ sitting in degree 0 and 1, where $\mathcal{E}_i$ is a vector bundle on $\mathrm{Spec}(A)$. Then the function

$$\chi_{\mathcal{L}} : \mathrm{Spec}(A) \to \mathbb{Z}, \;\; x \mapsto \chi(\mathcal{E}^0_x \to \mathcal{E}^1_x) = \mathrm{rk}(\mathcal{E}^0_x) - \mathrm{rk}(\mathcal{E}^1_x)$$

is locally constant. We let $\deg(\mathcal{L}) := \chi_{\mathcal{L}} + (g-1)$ be the desired locally constant function, thus defining the map $\deg : \mathrm{Bun}_{\mathbb{G}_m} \to \mathbb{Z}$.[4]

Next, let us describe the "shape" of $\mathrm{Bun}_{\mathbb{G}_m}$. Let $B\mathbb{G}_m$ be the stack such that $B\mathbb{G}_m(A)$ is the groupoid of line bundles on $\mathrm{Spec}(A)$. Recall that $B\mathbb{G}_m$ admits a smooth cover by the point scheme $\mathrm{pt} = \mathrm{Spec}(k)$. We have a canonical map

$$B\mathbb{G}_m \to \mathrm{Bun}_{\mathbb{G}_m}, \;\; \mathcal{L} \mapsto \mathcal{O}_X \boxtimes \mathcal{L}.$$

Fixing $x_0 \in X(k)$, we get a map in the other direction:

$$\mathrm{Bun}_{\mathbb{G}_m} \to B\mathbb{G}_m, \;\; \mathcal{L} \mapsto \mathcal{L}_{x_0}.$$

Let $\mathrm{Bun}^n_{\mathbb{G}_m} = \deg^{-1}(n)$. We define $\mathrm{Jac}(X)$ by requiring we have a Cartesian diagram

$$\begin{tikzcd} \mathrm{Jac}(X) \arrow[r] \arrow[d] & \mathrm{Bun}^0_{\mathbb{G}_m} \arrow[d] \\ \mathrm{pt} \arrow[r] & B\mathbb{G}_m \end{tikzcd}$$

These maps combine to give isomorphisms:

$$\mathrm{Bun}_{\mathbb{G}_m} \simeq B\mathbb{G}_m \times \mathrm{Jac}(X) \times \mathbb{Z},$$

$$\underline{\mathrm{Pic}} \simeq \times \mathrm{Jac}(X) \times \mathbb{Z}.$$

Fact: $\mathrm{Jac}(X)$ is an abelian variety (i.e., a geometrically connected smooth proper algebraic group).

Note that we have a canonical[5] Abel-Jacobi map:

$$X \to \mathrm{Bun}^1_{\mathbb{G}_m} \to \mathrm{Bun}_{\mathbb{G}_m}, \;\; x \mapsto \mathcal{O}(x).$$

In line with the previous subsection, the Abel-Jacobi map induces an isomorphism:

$$\pi_1^{\text{ét}}(X)^{\text{ab}} \xrightarrow{\simeq} \pi_1^{\text{ét}}(\mathrm{Bun}^1_{\mathbb{G}_m}).$$

[3]We remind that the latter means there exists a scheme $\widetilde{\mathrm{Pic}}$ such that for all affine schemes $S = \mathrm{Spec}(A)$, we have a bijection of sets $\mathrm{Hom}_{\mathrm{Schemes}}(S, \widetilde{\mathrm{Pic}}) = \underline{\mathrm{Pic}}(A)$, functorial in S.

[4]We add $(g-1)$ to $\chi_{\mathcal{L}}$ to normalize the degree map such that the trivial line bundle has degree 0.

[5]In the sense that it does not depend on a base point.

5 More on Curves: 16/09/2025

Scribe: David Fang

Let X be a smooth curve over k. Remember that a **divisor** D is a formal finite sum

$$\sum_{x\in X} n_x[x], \qquad n_x \in \mathbb{Z}, x \in X \text{ closed point.}$$

An **effective divisor** is a divisor where $n_x \geq 0, \forall x$. Then we have the

Fact 1. *There is a bijection:*

$$\{D \text{ divisor}\} \longleftrightarrow \{\mathcal{L} \text{ line bundle on with } s : \mathcal{O}_U \xrightarrow{\simeq} \mathcal{L}|_U, U \subsetneq X \text{ dense}\}/\sim,$$

where $\sim$ is up to isomorphisms of $\mathcal{L}$ and "shrinking U." Explicitly, for each divisor D we can associate the pair $(\mathcal{O}(D), s = 1)$. The reverse direction is obtained by sending $(\mathcal{L}, s)$ to $\div s$.

In particular, there is a bijection between effective divisors and pairs $(\mathcal{L}, s \in \Gamma(\mathcal{L})$, such that $s|_U \neq 0$ for some dense $U \subseteq X$. We want to generalize this to the situation of "curves over rings" instead. In particular

Definition 7. *Let S be a scheme. An **(effective) Cartier divisor** on S is a pair $(\mathcal{L}, s)$ where $\mathcal{L}$ is a line bundle on S, and s is a trivialization of $\mathcal{L}|_U$ for $U \subset S$ dense (resp. $s \in \Gamma(\mathcal{L})$ trivializing $\mathcal{L}$ on some U).*

Remark 7. *When we say $U \subseteq S$ is **dense**, we mean that S is the minimal closed subscheme of S containing U (**schematically dense**).*

Definition 8. *Given $X, S/k$ as above, a **relative (effective) Cartier divisor** is an (effective) Cartier divisor on $X \times S$ such that U is universally dense (i.e. for all $T \to S$, the base change $U \times_S T \subseteq X \times T$ is dense). In the effective case, we also require that $\operatorname{coker} \mathcal{O}_{X\times S} \xrightarrow{s} \mathcal{L}$ is S-flat.*

Morally, this is supposed to be an S-family of Cartier divisors on X.

Remark 8. *For example, a relative effective divisor is dense in every fiber of S (namely, take T to be Spec of a field in the above).*

Example 9. *Suppose $X = \operatorname{Spec} A, X = \mathbb{A}^1$. Then $X \times S = \operatorname{Spec} A[t]$. Let $s = f(t) = \sum_{i=0}^n a_i t^i, a_i \in A$. This defines a Cartier divisor iff $(a_0, a_1, \dots, a_n) = A$, and effective iff $a_n \in A^\times$.*

Theorem 8. *Let X be a smooth projective curve over k. Then there is a scheme $\operatorname{Sym} X/k$ such that*

$$S \to \operatorname{Sym} X \iff \{D \text{ an effective relative Cartier divisor on } X \times S\}$$

Here $\operatorname{Sym} X = \coprod_{n\geq 0} \operatorname{Sym}^n X$, and $\operatorname{Sym}^0 X = \operatorname{Spec} k, \operatorname{Sym}^1 X = X$. More generally, there is a map $X^n \to \operatorname{Sym}^n X$, which identifies $X^n/S_n \simeq \operatorname{Sym}^n X$. Also, $\operatorname{Sym}^n X$ is smooth.

Example 10. *Let $X = \mathbb{A}^1$. Then $\operatorname{Sym} X \simeq \mathbb{A}^n$ as follows: for a point $(a_0, \dots, a_{n-1}) \in \mathbb{A}^n$, we can associate the divisor of zeroes of $t^n + a_{n-1}t^{n-1} + \cdots + a_0$.*

Exercise 1. $\operatorname{Sym}^n \mathbb{P}^1 = \mathbb{P}^n$.

Now return to the case where X is smooth projective. There's an obvious map

$$\mathrm{AJ}_n : \operatorname{Sym}^n X \to \mathrm{Bun}^n_{\mathbb{G}_m}, \qquad (\mathcal{L}, s) \to \mathcal{L}$$

Note that $\mathrm{AJ}_n^{-1}(\mathcal{L}) = \Gamma(\mathcal{L}) \setminus 0$.

Remark 9. *If we work with $\underline{\mathrm{Pic}}^n$ instead, then $\mathrm{AJ}_n^{-1}(\mathcal{L}) = \mathbb{P}(\Gamma(\mathcal{L})) = (\Gamma(\mathcal{L}) \setminus 0/\mathbb{G}_m$*

Claim 1. *For $n \gg 0$ (i.e. $n > 2g-2, g = g(X)$), then*

$$\mathrm{AJ}_n : \operatorname{Sym}^n X \to \mathrm{Bun}^n_{\mathbb{G}_m}$$

is smooth and surjective. In fact, it is locally a product in the smooth topology.

Proof. Say Riemann-Roch a bunch. More precisely, if $\mathcal{L} \in \mathrm{Bun}^n_{\mathbb{G}_m}$, then

$$H^i(\mathcal{L}) \simeq H^{1-i}(\mathcal{L}^\vee \otimes \Omega^1)^\vee$$

by Serre duality. Since $\deg \Omega^1 = 2g-2$, we know that if $\deg \mathcal{L} > 2g-2$, then $\deg \mathcal{L}^\vee \otimes \Omega^1 < 0$. Thus $H^0(\mathcal{L}^\vee \otimes \Omega^1 = 0$, so $H^1(\mathcal{L}) = 0$ by Serre duality. But by Riemann Roch,

$$\deg \mathcal{L} = \dim H^0(\mathcal{L}) - \dim H^1(\mathcal{L}) + g - 1 = \dim H^0(\mathcal{L}) + g - 1,$$

so when $\deg \mathcal{L} \gg 0$, we have

$$\dim H^0 \mathcal{L} = \underbrace{\deg \mathcal{L}}_{n} + 1 - g.$$

We in fact have that

$$\{\mathcal{L} \in \mathrm{Bun}^n_{\mathbb{G}_m} + s \in \mathcal{L}\} \to \mathrm{Bun}^n_{\mathbb{G}_m}$$

is the total space of a vector bundle, given by the pushforward of the universal line bundle on $X \times \mathrm{Bun}^n_{\mathbb{G}_m}$ to $\mathrm{Bun}^n_{\mathbb{G}_m}$ $(n \gg 0)$. □

Remark 10. *In general (without assumption on* n*) the space* $\{\mathcal{L} \in \mathrm{Bun}^n_{\mathbb{G}_m} + s \in \mathcal{L}\}$ *will be the total space of a coherent sheaf (derived if you want) over* $\mathrm{Bun}^n_{\mathbb{G}_m}$*.*

Upshot: there is a map

$$\mathrm{Sym}^n X \to \mathrm{Bun}^n_{\mathbb{G}_m}$$

whose fibers are $\mathbb{A}^N \backslash 0$ $(N = n + 1 - g)$. In particular the fibers become more and more contractible as $n \to \infty$. Alternatively, after picking a base point $x_0 \in X$ the map

$$\mathrm{Sym}^n X \to \mathrm{Jac}\, X$$

has fibers isomorphic to $\mathbb{P}^{N-1}$.

Remark 11 (A rough analogy)**.** *For a manifold* M*, Dold-Thom says roughly that if we take* $\mathrm{Sym}^n M$ *for sufficiently large* n*, then its homotopy groups in low degree are isomorphic to the homology groups of* M*.*

6 Strategy for GCFT: 18/09/2025

Scribe: Soumik Ghosh

Exercise: $U = \operatorname{Spec} A[f^{-1}] \subset S = \operatorname{Spec} A$ is dense $\iff$ f is a non-zero divisor.

Want to prove: We have the isomorphism

$$\pi_1^{ab}(X) \xrightarrow{\simeq} \pi_1^{et}(\operatorname{Jac} X)$$
$$\iff \pi_1^{ab}(X) \xrightarrow{\simeq} \pi_1^{et}(\operatorname{Bun}^1_{\mathbb{G}_m})$$

Remark 12. *Th etale fundamental group does not generally satisfy Künneth in* $\operatorname{char} p > 0$. *But, it is true for proper varieties/*$k = \bar{k}$

Motto: enemy= wild ramification at ∞.

If A is an abelian variety (smooth, connected, proper group scheme), $\pi_1^{et}(A)$ is abelian.

Idea: We have the group law $m : A \times A \to A$ induces a commutative diagram

$$\begin{tikzcd}
\pi_1^{et}(A \times A) \arrow[r, "m"] \arrow[d, "pr_1 \times pr_2"', "\simeq"] & \pi_1^{et}(A) \\
\pi_1^{et}(A) \times \pi_1^{et}(A) \arrow[ur, "\text{check this is the usual group structure}"'] &
\end{tikzcd}$$

6.1 Structure of $\pi_1^{ab}(X)$

π_1^{et} is a pro-finite group.

π_1^{ab} is pro-(finite, abelian) group.

$\implies \pi_1^{ab} \simeq \prod_{l \text{ prime}} (\pi_1^{ab})_l$ a product of pro-abelian l-groups.

Remark 13. $l \neq p$, $(\pi_1^{ab})_l$ *behaves like* $H_1(\bullet, \mathbb{Z}_l)$ *(it is:* $H_1^{et}(\bullet, \mathbb{Z}_l)$*)*

If $l = p$, *then it is more complicated.*

Example 11. $(\pi_1^{ab}(\mathbb{A}^1))_l = \begin{cases} 0 \text{ if } l \neq p \\ \text{infinite if } l = p \end{cases}$

6.2 Strategy for GCFT (after Deligne)

We prove: $\operatorname{Hom}(\pi_1^{et}(\operatorname{Jac}), e^\times) \simeq \operatorname{Hom}(\pi_1^{et}(X), e^\times)$ where $e = \bar{\mathbb{Q}}_l$ where $l \neq p$ or $e =$ finite extension of $\mathbb{Q}_l$.

It suffices by Pontryagin Duality to prove this assertion.

Recall from topology: If M is a connected manifold and we have a group homomorphism $\pi_1(M) \xrightarrow{\rho} \operatorname{GL}_n(e)$ where e is a commutative ring, then we have the bijection

$$\left\{\pi_1(M) \xrightarrow{\rho} \mathrm{GL_n}(e)\right\} \leftrightarrow \{\text{local system of rank } n \text{ free } e\text{-modules}\}$$

There is a theory of l-adic (etale) sheaves in algebraic geometry.

Fix Λ, a commutative ring such that it is $\in \begin{cases} \text{finite of order prime to } p \\ \mathcal{O}_e,\ e/\mathbb{Q}_l \text{ algebraic extension} \\ e/\mathbb{Q}_l \text{ algebraic extension} \end{cases}$

For all Y, Noetherian scheme, we have $\operatorname{lisse}(Y, \Lambda) \subset \operatorname{Shv}(Y, \Delta)$.

We write $\operatorname{Shv}(Y, \Lambda)^\heartsuit$ for the abelian category version.

If Y is connected, $\operatorname{Lisse}(Y, \Lambda)^\heartsuit = \operatorname{Rep}^\Lambda_{cont}(\pi_1^{et}(Y, y))$ ie continuous representations of $\pi_1^{et}(Y, y)$ on Λ-modules.

We have functors f^*, f_* for Shv.

Fix e to be an algebraic extension of $\mathbb{Q}_l$.

We have the map

$$\{\operatorname{rk} 1 \text{ lisse sheaves on } \operatorname{Jac} X\} \xrightarrow{\text{restriction along AJ}} \{\operatorname{rk} 1 \text{ lisse sheaves on } X\}$$

Claim: This is an equivalence.
Strategy: Produce

$$\begin{tikzcd} \{\mathrm{rk}\,1 \text{ lisse sheaves on } X\} \arrow[r] \arrow[dr, "id"] & \{\mathrm{rk}\,1 \text{ lisse sheaves on } \mathrm{Bun}_{\mathbb{G}_m}\} \arrow[d, "\mathrm{AJ}^*"] \\ & \{\mathrm{rk}\,1 \text{lisse sheaves on } X\} \end{tikzcd}$$

Definition 9. *σ is lisse on $X \implies \sigma^{(n)} \in \mathrm{Shv}(\mathrm{Sym}^n X)$*

$$\sigma \boxtimes \cdots \boxtimes \sigma = \bigotimes_{i=1}^{n} pr_i^* \sigma \in \mathrm{Lisse}(X^n) \iff \pi_1(X)^n \to (\mathrm{GL_r})^n \to \mathrm{GL_{r^n}}$$

These are S_n-equivariant sheaves where S_n is the symmetric group.

We have

$$\mathrm{add}_{n*}(\sigma \boxtimes \cdots \boxtimes \sigma) \in \mathrm{Shv}(\mathrm{Sym}^n X)$$

where

$$\begin{aligned} \mathrm{add_n} : & X^n \to \mathrm{Sym}^n X \\ & (x_1, \ldots, x_n) \mapsto \sum_i [x_i] \end{aligned}$$

The S_n-equivariant structure becomes an action of S_n on $\mathrm{add}_{n*}(\sigma^{\boxtimes n})$. I take $\sigma^{(n)} := \mathrm{add}_{n*}(\sigma^{\boxtimes n})^{S_n}$

Explicitly: fiber of $\sigma^{\boxtimes n}$ at $(x_1, x_2, \ldots, x_n)$ is $\sigma_{x_1} \otimes \cdots \otimes \sigma_{x_n}$

fiber of $\mathrm{add}_{n*}(\sigma^{\boxtimes n})$ at D is $\bigoplus_{D=\sum_i x_i} \sigma_{x_1} \otimes \ldots \sigma_{x_n}$ and the action of S_n changes the ordering. Say $D = [x] + [y]$ with $x \neq y$. Then the fiber is $\sigma_x \otimes \sigma_y \oplus \sigma_y \otimes \sigma_x \implies$ space of invariants is isomorphic to $\sigma_X \otimes \sigma_y$. IF $D = 2[x]$, then the fiber is $\sigma_x \otimes \sigma_x$, so the space of invariants is $\simeq \mathrm{Sym}^2 \sigma_x$.

Say $D = \sum_{x_i \neq x_j} n_i x_j$. So we get the invariant to be $\otimes_i \mathrm{Sym}^{n_i} \sigma_{x_i}$ Special Case: $r = 1$, then $\mathrm{Sym}^n \sigma_x$ also has rank 1 and $\sigma_X^{\otimes n} = \mathrm{Sym}^n X$. So $\sigma^{(n)}$ is a rk 1 local system with fiber at $D = \sum_i n_i x_i$ being $\otimes_i \sigma_{x_i}^{\otimes n_i}$

$\sigma^{(n)}$ on $\mathrm{Sym}^n X$ and for $n \gg 0$ descends to $\mathrm{Bun}^n_{\mathbb{G}_m}$ along AJ_n.

7 Towards Hecke Eigensheaves: 25/09/2025

Scribe: Youseong Lee

7.1 Recollection

Given a rank 1 local system σ on X, we want a rank 1 "special" local system χ_σ on $\mathrm{Bun}_{\mathbb{G}_m}$ such that $AJ^*(\chi_\sigma) = \sigma$. Last time, we constructed a rank 1 local system on $\mathrm{Sym}^n X$

$$\sigma^{(n)} := add_{n*}(\sigma \boxtimes \ldots \boxtimes \sigma)^{S_n}$$

whose fiber on a divisor $D = \sum_i n_i x_i$ is $\sigma_{x_i}^{\otimes n_i}$. A little more formally, one can say:

Proposition 3 (Exercise)**.** *The canonical map $\sigma \boxtimes \ldots \boxtimes \sigma \to add_n^*(\sigma^{(n)})$ given by adjunction is an isomorphism. (Easy to check!)*

Remark 14. *When* $\operatorname{rank}\sigma > 1$*, there may be some extra complication.*

Observation from the end of Lecture 5: for $n > 2g-2$,

$$\mathrm{Sym}^n X \xrightarrow{AJ_n} \mathrm{Bun}^n_{\mathbb{G}_m}$$
$$\mathrm{Sym}^n X \to \underline{\mathrm{Pic}}^n$$

have fibers $\mathbb{A}^N \backslash 0$ and $\mathbb{P}^{N-1}$, respectively, for $N = n+1-g$. It follows that $\sigma^{(n)}$ is constant along these fibers, so it descends to $\chi^n_\sigma \in Lisse(\mathrm{Bun}^n_{\mathbb{G}_m})$ for $n > 2g-2$.

If you like:

1. One can think χ^n_σ as the non-derived pushforward of $\sigma^{(n)}$.
2. Given a smooth surjective map $f : X \to Y$ between smooth X, Y with geometrically connected fibers, the pullback $Lisse(Y)^\heartsuit \to Lisse(X)^\heartsuit$ is fully faithful with essential image as those local systems that are constant along fibers.
3. The above generalizes to perverse sheaves as well.

7.2 Digression

Checking the following proposition:

Proposition 4. *$\mathbb{P}^N$ is simply connected over algebraically closed field $k = \bar{k}$.*

Here, being simply connected means $\pi_1^{\acute{e}t}(\mathbb{P}^N) \simeq *$, or, $Lisse(\mathbb{P}^N)^\heartsuit \simeq \mathrm{Vect}_k^\heartsuit$.

Proof. First, we prove for $N = 1$.

Claim 2. *If $f : Y \to \mathbb{P}^1$ is finite étale map and Y is connected, then f is an isomorphism.*

Proof. By Riemann-Hurwitz formula,

$$\deg(f) \cdot \chi(\mathbb{P}^1) = \chi(Y)$$

holds. Now $\chi(\mathbb{P}^1) = 2$ and $\chi(Y) = 2 - 2g_Y$, so we have $LHS \geq 2$ and $RHS \leq 2$. Therefore we must have equality on both sides $LHS = RHS = 2$, so that $\deg f = 1$ and f is an isomorphism. ☐

The claim completes the proof for $N = 1$; it asserts that $\mathbb{P}^1$ allows no further étale covering maps.

For general N, we need two inputs from SGA I:

1. Künneth formula for proper varieties:

$$\pi_1^{\acute{e}t}\left(\mathbb{P}^1 \times \ldots \times \mathbb{P}^1\right) = *.$$

2. For smooth Y and $Z \subseteq Y$ of codimension ≥ 2,

$$\pi_1^{\acute{e}t}(Y \backslash Z) \xrightarrow{\sim} \pi_1^{\acute{e}t}(Y).$$

As a corollary, we have:

Corollary 2. *$\pi_1^{\acute{e}t}$ is a birational invariant for smooth proper varieties.*

Since $\mathbb{P}^N$ is birationally equivalent to $\left(\mathbb{P}^1\right)^N$, we have $\pi_1^{\acute{e}t}(\mathbb{P}^N) = *$ as desired. $\square$

7.3 Multiplicative sheaves

Choose some $\mathcal{L}_0$ with $\deg \mathcal{L}_0 >> 0$. For all n,

$$\chi_\sigma^n = \chi_\sigma^{n+N\cdot\deg\mathcal{L}_0} \otimes \chi_\sigma|_{\mathcal{L}_0}^{\otimes -N}.$$

Here, in the first equality, we used $\mathcal{L}_0$ to identify

$$\mathrm{Bun}_{\mathbb{G}_m}^n \xrightarrow{\sim} \mathrm{Bun}_{\mathbb{G}_m}^{n+\deg\mathcal{L}_0} \xrightarrow{\sim} \ldots \xrightarrow{\sim} \mathrm{Bun}_{\mathbb{G}_m}^{n+N\cdot\deg\mathcal{L}_0}$$

We choose the minimal N such that $n + N \cdot \deg \mathcal{L}_0 > 2g - 2$.

We want to express this a little scientifically!

Let A be a commutative group scheme (which will be $\mathrm{Jac}(X)$ in the future,) and e is the coefficient field for our sheaves.

Definition 10. *A multiplicative sheaf on A is a local system $\chi \in Lisse(A)$ with data*

$$1^*\chi \simeq e$$
$$m^*\chi \simeq \chi \boxtimes \chi$$

where $1 : \mathrm{Spec}\, k \to A$, $m : A \times A \to A$ are the unit map and multiplication, respectively. These isomorphisms should satisfy various "obvious" axioms:

1. *they are compatible morphisms;*
2. *the second isomorphism is $\mathbb{Z}/2$-equivariant;*
3. *they satisfy the cocycle condition on A^3.*

Remark 15. *The isomorphism $m^*\chi \simeq \chi \boxtimes \chi$ is analogous to the property of characters, in the sense that on the fiberwise level we have $\chi_{ab} \simeq \chi_a \otimes \chi_b$. (Note that the axioms force χ to have rank 1.)*

Example 12. *Consider a finite covering of A given as:*

$$0 \to \Gamma \to \widetilde{A} \to A \to 0$$

with finite group Γ and a commutative group scheme $\widetilde{A}$, together with a character $\Gamma \to e^\times$. (A good example is $(-)^n : \mathbb{G}_m \to \mathbb{G}_m$ with $\Gamma \simeq \mathbb{Z}/n$.) Then we get

$$\pi_1^{\acute{e}t}(A) \xrightarrow{\textit{classifying cover}} \Gamma \to e^\times$$

which gives a rank 1 local system χ on A; it will be multiplicative.

We want to establish the following correspondence:

$$\{\text{Multiplicative sheaves on } \mathrm{Bun}_{\mathbb{G}_m}\} \xrightarrow{\sim} \{\text{Rank 1 local systems on } X\}$$
$$\chi \mapsto AJ^*(\chi).$$

Choose $x_0 \in X(k)$ for convenience to get

$$\mathrm{Bun}_{\mathbb{G}_m} \simeq B\mathbb{G}_m \times \mathrm{Jac}(X) \times \mathbb{Z}.$$

Claim 3. *There is isomorphism*

$$\{\textit{Multiplicative sheaves on } \mathrm{Bun}_{\mathbb{G}_m}\} \simeq \{\textit{Multiplicative sheaves on } \mathrm{Jac}\, X\} \times \{\textit{Lines}\}$$

that maps a multiplicative sheaf χ on $\mathrm{Bun}_{\mathbb{G}_m}$ to

1. *χ_{Jac}, the multiplicative sheaf on $\mathrm{Jac}(X)$ given by restriction of χ along $\mathrm{Jac}(X) \to \mathrm{Bun}_{\mathbb{G}_m}$;*
2. *and the line (1-dimensional e-vector space) given by tanking fiber on $\mathcal{O}(x_0)$.*

Here, the point $\mathcal{O}(x_0) \in \mathrm{Bun}_{\mathbb{G}_m} \simeq B\mathbb{G}_m \times \mathrm{Jac}(X) \times \mathbb{Z}$ can be interpreted as $(unit, unit, 1)$. Also:

$$\left\{\begin{matrix}\textit{Multiplicative sheaves on} \\ \mathrm{Jac}\, X\end{matrix}\right\} \simeq \left\{\begin{matrix}\textit{Rank 1 local system } \chi \textit{ on } \mathrm{Jac}(X) \\ \textit{with isomorphism } 0^*(\chi) \simeq e\end{matrix}\right\}$$

Proof. Write $A = \mathrm{Jac}(X)$. The LHS is equivalent to a map $\pi_1^{\acute{e}t}(A) \to e^\times$ such that the following diagram commutes:

$$\begin{tikzcd}
\pi_1^{\acute{e}t}(A) \arrow[r] & e^\times \\
\pi_1^{\acute{e}t}(A) \times \pi_1^{\acute{e}t}(A) \arrow[u, "m"] \arrow[r] & e^\times \times e^\times \arrow[u, "\mathrm{mult}"']
\end{tikzcd}$$

and the RHS is equivalent to a map $\pi_1^{\acute{e}t}(A) \to e^\times$. Now the claim follows from the Künneth formula and Eckmann-Hilton argument. ☐

7.4 Hecke eigensheaves

For each $n \geq 0$, define $\alpha_n : \mathrm{Sym}^n X \times \mathrm{Bun}_{\mathbb{G}_m} \to \mathrm{Bun}_{\mathbb{G}_m}$ as $\alpha_n(D, \mathcal{L}) = \mathcal{L}_D$. These are compatible with summation of divisors $\mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{m+n} X$, i.e. the following diagram commutes:

$$\begin{tikzcd}
\mathrm{Sym}^n X \times \mathrm{Sym}^m X \times \mathrm{Bun}_{\mathbb{G}_m} \arrow[r, "\alpha_m"] \arrow[d] & \mathrm{Sym}^n X \times \mathrm{Bun}_{\mathbb{G}_m} \arrow[d, "\alpha_n"] \\
\mathrm{Sym}^{m+n} X \times \mathrm{Bun}_{\mathbb{G}_m} \arrow[r, "\alpha_{m+n}"] & \mathrm{Bun}_{\mathbb{G}_m} .
\end{tikzcd}$$

Definition 11. *Given a rank 1 local system σ on X, a Hecke eigensheaf on $\mathrm{Bun}_{\mathbb{G}_m}$ with eigenvalue σ is a sheaf $\mathcal{F}$ on $\mathrm{Bun}_{\mathbb{G}_m}$ equipped with isomorphisms*

$$\alpha_n^* \mathcal{F} \simeq \sigma^{(n)} \boxtimes \mathcal{F}$$

for each $n \geq 0$, which satisfy obvious compatibility conditions:

1. *For $n = 0$, it gives the identity isomorphism;*
2. *For $n, m \geq 0$, the isomorphisms are compatible with maps $\mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Sym}^{m+n} X$ and pullbacks $\sigma^{(m+n)} \mapsto \sigma^{(n)} \boxtimes \sigma^{(m)}$ in evident ways.*

Example 13. *For $n = 1$, the isomorphism induced by $\alpha_1^* \mathcal{F} \simeq \sigma^{(1)} \boxtimes \mathcal{F}$ on the fiber over $x \in X$ and $\mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m}$ reads*

$$\mathcal{F}|_{\mathcal{L}_x} \simeq \sigma_x \otimes \mathcal{F}|_{\mathcal{L}}.$$

Note that this definition of Hecke eigensheaves makes sense for sheaves defined only on $\bigsqcup_{n \geq N} \mathrm{Bun}^n_{\mathbb{G}_m}$ for some N: we can use

$$\mathrm{Sym}^n X \times \mathrm{Sym}^m X \to \mathrm{Bun}^{n-m}_{\mathbb{G}_m}$$

that maps $(D_1, D_2) \mapsto \mathcal{O}(D_1 - D_2)$. Following Hecke property, the restriction of $\mathcal{F}$ along this map is

$$\mathcal{F}|_{\mathcal{O}_X} \otimes \sigma^{(n)} \boxtimes \left(\sigma^{(m)}\right)^{-1}.$$

In this regard, we have the following:

$$\left\{\begin{matrix}\text{Hecke eigensheaves for } \sigma \\ \text{on } \mathrm{Bun}_{\mathbb{G}_m}\end{matrix}\right\} \simeq \left\{\begin{matrix}\text{Hecke eigensheaves for } \sigma \\ \text{on } \bigsqcup_{n \geq N} \mathrm{Bun}^n_{\mathbb{G}_m}\end{matrix}\right\}.$$

Essentially, we only need data on a single $\mathrm{Bun}^n_{\mathbb{G}_m}$ together with an identification with some other $\mathrm{Bun}^m_{\mathbb{G}_m}$.

Definition 12. *A normalized Hecke eigensheaf on* $\mathrm{Bun}^n_{\mathbb{G}_m}$ *is one with extra data of an isomorphism* $\mathcal{F}|_{\mathcal{O}_X} \simeq e$.

Hence we are to establish the following correspondences:

$$\begin{tikzcd}
\left\{\begin{matrix}\text{Multiplicative sheaves} \\ \text{on } \mathrm{Bun}_{\mathbb{G}_m}\end{matrix}\right\} \arrow[rr, "\sim"] \arrow[dr, "\sim", "AJ^*"'] & & \left\{\begin{matrix}\text{Rank 1 local systems } \sigma \\ + \text{ Normalized Hecke eigensheaves} \\ \text{on } \mathrm{Bun}_{\mathbb{G}_m}\end{matrix}\right\} \arrow[dl, "\sim"] \\
& \left\{\begin{matrix}\text{Rank 1 local systems} \\ \text{on } X\end{matrix}\right\} &
\end{tikzcd}$$

8 Back to $\mathbb{F}_q$: 30/09/2025

Scribe: Michael Horzepa

The theme of this lecture is trying to gain arithmetic information out of geometric information, and the workhorse for this will be the **Lang isogeny**. We will first introduce some necessary construction before we state and give a basic proof of Lang's theorem, then we will move onto some applications.

8.1 Lang Isogeny

Begin with a group scheme H over $\mathbb{F}_q$. Then there exists a canonical finite subgroup $H(\mathbb{F}_q) \subseteq H$ by taking the $\mathbb{F}_q$ points of H. Any algebraist would then naturally take the quotient of H by this subgroup, leading to the question:

Question: What is $H/H(\mathbb{F}_q)$?

Answer: H if H is connected.

Counterexample: If H is discrete (say a finite group considered as a group scheme over $\mathbb{F}_{\shortparallel}$), then this quotient can become trivial, or more generally create a distinct quotient group.

To see this, we begin with some constructions: the absolute and geometric Frobenius. For the former, let us begin with $S = \operatorname{Spec} A$ for some finite $\mathbb{F}_q$ algebra A. Then we can produce a natural Frobenius map on A via

$$A \to A \qquad f \mapsto f^q$$

This then produces a map

$$\varphi : S \to S$$

By Functoriality this generalizes to any scheme over $\operatorname{Spec} \mathbb{F}_q$, and we call this map the "Absolute Frobenius".

Remark 16. *If S is a scheme over $\mathbb{F}_q$, then given $s \in S(B)$ for some algebra $B/\mathbb{F}_q$ we see that $\varphi(s)$ will give the image under the qth power map. Furthermore, using φ on B, we induce a map*

$$S(B) \to S(B)$$

Now we would like to introduce a variant of this Frobenius map, and to start we need $S_0/\mathbb{F}_q$ a scheme with base change $S/\overline{\mathbb{F}_q}$. Then we define the "Geometric Frobenius" via the morphsim of schemes over $\overline{\mathbb{F}_q}$ given by:

$$\Phi_S = \varphi_{S_0} \times id_{\mathbb{F}_q} : S_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q} \to S_0 \times_{\mathbb{F}_q} \overline{\mathbb{F}_q}$$

Example 14. *Let $S_0 = \{y^2 = x^3 + \lambda\} \in \mathbb{A}^2_{\mathbb{F}_q}$ and $S = \{y^2 = x^3 + \lambda\} \in \mathbb{A}^2_{\overline{\mathbb{F}_q}}$, where $\lambda \in \mathbb{F}_q$. Then*

$$\Phi_S(x, y) = (x^q, y^q)$$

Both of these maps will be major players, and a motto to take forward is that the "absolute Frobenius is great, geometric (Frobenius) is even better". Now with these defined, we return back to dealing with $H_0/\mathbb{F}_q$ a group scheme with $H/\overline{\mathbb{F}_q}$ a base change.

Definition 13. *The Lang Isogeny is either of the following two maps:*

$$H_0 \xrightarrow{\mathcal{L}} H_0 \qquad h \mapsto h\varphi(h)^{-1}$$
$$H \xrightarrow{\mathcal{L}} H \qquad h \mapsto h\Phi_H(h)^{-1}$$

We notice that this is functorial, and hence is a group homomorphism. Furthermore, for any point $h \in H_0(\mathbb{F}_q)$, we see that since points in $\mathbb{F}_q$ are fixed by the Frobenius, this means $\varphi(h) = h$ and $\mathcal{L}(h) = 1$. More generally, if we consider any element $\gamma \in H$, we see:

$$\mathcal{L}(\gamma h) = \gamma h \Phi_H(h)^{-1} \Phi_H(\gamma)^{-1} = \gamma \Phi_H(\gamma)^{-1} = \mathcal{L}(\gamma)$$

and hence $\mathcal{L}$ is invariant on the cosets of $H_0(\mathbb{F}_q)$. We can now come to our theorem:

Theorem 9. *(Lang's Theorem) If H_0 is connected, then $\mathcal{L}$ induces an isomorphism of left H_0 space*

$$H_0/H_0(\mathbb{F}_q) \xrightarrow{\simeq} H_0$$

To prove this theorem, we quickly need to address the left action of H_0 on itself. This action will be called "Frobenius conjugation" and acts via:

$$h * \gamma = h\gamma\varphi(h)^{-1}$$

With this in hand, the following lemma will for the most part immediately imply the theorem:

Lemma 2. *Every orbit of this action is open*

Proof. Let $\gamma \in H_0$, and build the map $H_0 \to H_0$ via $h \mapsto h * \gamma$. This gives an isomorphism of the tangent spaces because φ has 0-derivative, which proves the lemma. □

Now to prove Lang's theorem:

Proof. If H_0 is connected, then the lemma implies that there is exactly 1 H_0-orbit of this action, or put another way, that this action is transitive. Thus we have

$$H_0/\operatorname{Stab}(1) \xrightarrow{\simeq} H_0$$

The proof then finishes by realizing that $\operatorname{Stab}(1)$ is exactly $H_0(\mathbb{F}_q)$ □

Let us give a few brief examples:

Example 15. *For the most trivial example, consider $\mathbb{A}^1_{\mathbb{F}_q}$ or $\mathbb{A}^1_{\overline{\mathbb{F}_q}}$ thought of us as $\mathcal{G}_a$, the Lang isogeny is given by:*

$$\mathcal{L} : t \mapsto t - t^q$$

Both are infinite in any sense of the word, and by quotienting by this finite discrete set, they remain unchanged.

Remark 17. *Given an algebra $K/\mathbb{F}_q$ and $\lambda \in K$, by setting $K' = K[t]/(t - t^q - \lambda)$, we induce an Artin-Scheier cover through the Lang isogeny in the following way:*

$$\begin{tikzcd}
\operatorname{Spec} K' \arrow[r] \arrow[d] & \mathbb{A}^1 \arrow[d, "\mathcal{L}"] \\
\operatorname{Spec} K \arrow[r] & \mathbb{A}^1
\end{tikzcd}$$

Example 16. *This is the most trivial example. Let us consider $\mathcal{L} : \mathcal{G}_m \to \mathcal{G}_m$. This acts via $t \mapsto t^{q-1}$. This is an étale cover because $q - 1$ is prime to p, and hence via the theory of deck transformations we get associated Galois group*

$$\mathcal{H}_{q-1} = \mathbb{F}_q^\times = \mathcal{G}_m(\mathbb{F}_q)$$

8.2 Applications

Let us run through several applications of this theorem:

Example 17. *A corollary of Lang's theorem is the following:*

Corollary 3. *If H_0 is connected, then any H_0-torsor over $\operatorname{Spec}\mathbb{F}_q$ is trivial. Here we have $H_0 \curvearrowright P \to \operatorname{Spec}\mathbb{F}_q$. Then $P|_{\mathbb{F}_q} \simeq H$ in a manner compatible with the action, which also is true if and only if $P(\mathbb{F}_q) \neq \emptyset$.*

Proof. We know $\exists p \in P(\mathbb{F}_q)$ by assumption. Apply $\Phi_P(p) = h \cdot p$ for some $h \in H(\overline{\mathbb{F}_q})$. Then by Lang there exists $\gamma \in H$ such that $\gamma^{-1}\Phi_H(\gamma) = h$. Taking then $p^{new} = \gamma \cdot p$, we get

$$\begin{aligned} \Phi_H(p^{new}) &= \Phi_H(\gamma) \cdot \Phi_H(p) && (15) \\ &= (\gamma h^{-1})(hp) = \gamma p = p^{new} && (16) \end{aligned}$$

This then implies $p^{new} \in P(\mathbb{F}_q)$, so we are done (using the comment about the equivalent condition). □

Example 18. *Now we return to a smooth geometrically connected curve $X_0/\mathbb{F}_q$. Then we Lang implies that there exists a divisor $\mathcal{D}$ of degree 1 on X_0. To see this, let $H_0 = \underline{Pic}^0 \curvearrowright P = \underline{Pic}^1$. This then implies the existence of a line bundle of degree 1 trivialized on some $\mathcal{U} \subseteq X_0$, which gives rise to our divisor.*

Exercise 2. *As an alternative proof, use the Weil bound to deduce this same conclusion.*

Example 19. *Lang provides a way to show that every finite dimensional division algebra $\mathcal{D}/\mathbb{F}_q$ is commutative.*

Proof. Without loss of generality, assume $\mathcal{D}$ is central over $\mathbb{F}_q$ of rank n. Then the result follows by letting $H_0 = PGL_n = \mathrm{Aut}_{\mathrm{Alg}}(\mathcal{M}_n)$ and $P = \{\mathcal{D} \simeq \mathcal{M}_\backslash\}$, where $\mathcal{M}_n$ is the matrix algebra. □

Example 20. *Let G be a connected reductive group scheme over $\mathbb{F}_q$. Then we can construct the flag variety*

$$F\ell_G = \{B \subset G \ \textit{Borel}\}$$

By the general theory $G \curvearrowright F\ell_G$ transitively via conjugation. Using Lang we can then deduce that we actually have some $B \subset G$ which is defined over $\mathbb{F}_q$.

Example 21. *For the last example, let $p \neq 2$. Then given a quadratic form $q = \{\sum_{i=1}^n a_i x_i^2 | a_i \neq 0, \prod a_i \in (\mathbb{F}_q^\times)^2\}$, then q is equivalent to $\sum y_i^2$ up to a choice of coordinates.*

Proof. This follows from the equivalence

$$\left\{ \begin{array}{c} \text{Non-deg. rank n quad.} \\ \text{forms w/ disc. in } k^\times/(k^\times)^2 \end{array} \right\} \leftrightarrow \{SO_n - \text{torsors}\}$$

With the latter being trivial, the result follows. □

From all this discussion, what we really need is for our group scheme $H_0/\mathbb{F}_q$, we get a canonical map (and further a group homomorphism)

$$\pi_1^{\grave{e}t}(H) \twoheadrightarrow H(\mathbb{F}_q)$$

which is encoded by the connected cover $H_0 \xrightarrow{\mathcal{L}} H_0$. One might recall this is a similar sort of induced map that comes from regular topological fundamental groups and connected covers.

Remark 18. *We can see $\mathbb{Z}/(n) \curvearrowright \mathbb{F}_{q^n}$ by $t \to t^q$. Then this implies $\mathbb{Z}/(n) \curvearrowright \operatorname{Spec}\mathbb{F}_{q^n} \to \operatorname{Spec}\mathbb{F}_q$ is a nontrivial $\mathbb{Z}/(n)$-torsor.*

9 GCFT via Frobenius: 02/10/2025

Scribe: Vladyslav Zveryk

9.1 Recap: Frobenius and Rational Points

Let Y_0 be a variety defined over a finite field $\mathbb{F}_q$. Let $Y = Y_0 \times_{\operatorname{Spec}\mathbb{F}_q} \operatorname{Spec}\bar{\mathbb{F}}_q$ be its base change to the algebraic closure. The **geometric Frobenius** is the morphism $\Phi : Y \to Y$ which is the base change of the map om Y_0 which is identity on the topological space and the q-th power map on the structure sheaf. For an affine variety $Y_0 = \operatorname{Spec}\mathbb{F}_q[x_1, \ldots, x_n]/(f_i)$, the Frobenius acts on coordinates as $\Phi(y_1, \ldots, y_n) = (y_1^q, \ldots, y_n^q)$.

This setup leads to a fundamental dictionary:

$$\left\{ \begin{array}{c} \text{Arithmetic geometry of } Y_0 \\ \text{over } \mathbb{F}_q \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{Geometry of } Y \text{ over } \bar{\mathbb{F}}_q \\ \text{together with the Frobenius } \Phi \end{array} \right\}.$$

The set of $\mathbb{F}_q$-rational points of Y_0, denoted $Y_0(\mathbb{F}_q)$, can be realized as the set of fixed points of the Frobenius map on Y:

$$Y_0(\mathbb{F}_q) = Y^\Phi(\overline{\mathbb{F}}_q).$$

Geometrically, the fixed points are the intersection of the graph of the Frobenius Graph(Φ) with the diagonal morphism Δ inside the product space $Y \times Y$.

$$\begin{tikzcd} Y^\Phi \arrow[r] \arrow[d] & Y \arrow[d, "\mathrm{Graph}(\Phi)"] \\ Y \arrow[r, "\Delta"] & Y \times Y . \end{tikzcd}$$

Note that Y^Φ is a finite scheme over $\overline{\mathbb{F}}_q$ whose $\overline{\mathbb{F}}_q$-points are naturally identified with the $\mathbb{F}_q$-points of Y_0.

Example 22. *Let $Y_0 = \mathbb{A}^1_{\mathbb{F}_q}$. Then $Y = \mathbb{A}^1_{\bar{\mathbb{F}}_q} = \operatorname{Spec}\bar{\mathbb{F}}_q[x]$. The fixed points are given by the spectrum of the ring:*

$$(\mathbb{A}^1_{\bar{\mathbb{F}}_q})^\Phi = \operatorname{Spec}\left(\frac{\bar{\mathbb{F}}_q[x] \otimes_{\bar{\mathbb{F}}_q} \bar{\mathbb{F}}_q[y]}{(x-y, x^q - y)}\right) \cong \operatorname{Spec}\left(\frac{\bar{\mathbb{F}}_q[y]}{y^q - y}\right)$$

Since $y^q - y = \prod_{a \in \mathbb{F}_q}(y-a)$, this corresponds to the disjoint union of points for each element of $\mathbb{F}_q$:

$$\operatorname{Spec}\left(\prod_{a \in \mathbb{F}_q} \frac{\bar{\mathbb{F}}_q[y]}{(y-a)}\right) \cong \coprod_{a \in \mathbb{F}_q} \operatorname{Spec}(\bar{\mathbb{F}}_q) \cong \mathbb{A}^1(\mathbb{F}_q)$$

Remark 19 (The case of stacks)**.** This principle extends to algebraic stacks. Let H_0 be an algebraic group over $\mathbb{F}_q$, and let $H = H_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Consider the classifying stack $Y_0 = \mathcal{B}H_0 = [*/H_0]$, whose base change is $Y = \mathcal{B}H = [*/H]$. The $\mathbb{F}_q$-points of the stack, $(\mathcal{B}H_0)(\mathbb{F}_q)$, correspond to isomorphism classes of H_0-torsors over $\operatorname{Spec}(\mathbb{F}_q)$.

The fixed-point stack Y^Φ can be described in terms of Φ-conjugacy classes on the group H. We have

$$Y^\Phi = \mathcal{B}H \times_{\mathcal{B}(H \times H)} \mathcal{B}H = H_\Delta \backslash H \times H /_\Phi H = H/_\Phi H,$$

where Δ is the diagonal action and the Φ-action is given by $g \mapsto hg\Phi(h)^{-1}$ for $g, h \in H(\bar{\mathbb{F}}_q)$.

A key result discussed last time is that every orbit under this Φ-conjugacy action is open. Therefore, if H is a connected group, there is a unique Φ-conjugacy class. In this case,

$$Y^\Phi = H/_\Phi H = [*/H_0(\mathbb{F}_q)] = \mathcal{B}(H_0(\mathbb{F}_q))$$

because $H_0(\mathbb{F}_q)$ is the stabilizer of the identity element under the Φ-conjugacy action. More concretely, we have a fiber product diagram:

$$\begin{tikzcd} H \arrow[r] \arrow[d, "\mathcal{L}"'] & * \arrow[d] \\ H \arrow[r] & \mathcal{B}(H_0(\mathbb{F}_q)), \end{tikzcd}$$

which makes $\mathcal{L}$ an $H_0(\mathbb{F}_q)$-torsor.

This implies that there is a unique H_0-torsor over $\mathrm{Spec}(\mathbb{F}_q)$ up to isomorphism. The group of automorphisms of this torsor is $H_0(\mathbb{F}_q)$.

9.2 Application to Geometric Class Field Theory

Let X_0 be a smooth, geometrically connected projective curve over $\mathbb{F}_q$, and let $X = X_0 \times_{\mathbb{F}_q} \bar{\mathbb{F}}_q$. Geometric class field theory studies the abelianization of the Weil group of X_0. The **Weil group** W_{X_0} is a subgroup of the arithmetic étale fundamental group $\pi_1^{\text{ét}}(X_0)$ that fits into the following short exact sequence:

$$1 \longrightarrow \pi_1^{\text{ét}}(X) \longrightarrow W_{X_0} \longrightarrow \mathbb{Z} \longrightarrow 0, \tag{17}$$

where $\pi_1^{\text{ét}}(X)$ is the geometric fundamental group of X and the $\mathbb{Z}$ is generated by the Frobenius automorphism.

The main goal of geometric class field theory is to provide a geometric description of the abelianization of the Weil group, $W_{X_0}^{ab}$. The fundamental isomorphism is:

$$W_{X_0}^{ab} \cong \mathrm{Pic}(X_0)$$

compatible with $W_{X_0}^{ab} \to \mathbb{Z}$ and the degree map $\mathrm{Pic}(X_0) \to \mathbb{Z}$.

If the curve X_0 has an $\mathbb{F}_q$-rational point, say $x_0 \in X_0(\mathbb{F}_q)$, this provides additional structure. The existence of x_0 gives a section $\mathrm{Spec}(\mathbb{F}_q) \to X_0$, which induces a map

$$\mathbb{Z} = W_{\mathrm{Spec}(\mathbb{F}_q)} \to W_{X_0}$$

splitting the sequence (17):

$$W_{X_0} \cong \pi_1^{\text{ét}}(X) \rtimes \mathbb{Z}$$

Similarly, the Picard group decomposes. The degree map $\deg : \mathrm{Pic}(X_0) \to \mathbb{Z}$ has a section given by x_0, leading to a decomposition:

$$\mathrm{Pic}(X_0) \cong \mathrm{Pic}^0(X_0) \times \mathbb{Z} \cong \mathrm{Jac}(X_0)(\mathbb{F}_q) \times \mathbb{Z}.$$

9.3 Frobenius Action and the Reciprocity Map

9.3.1 Frobenius Action on the Fundamental Group

One question one could ask is how does $\mathbb{Z}$ act on the geometric fundamental group $\pi_1^{\text{ét}}(X)$ in the semidirect product decomposition of W_{X_0}. It turns out that this action is given by a canonical action of the geometric Frobenius Φ on $\pi_1^{\text{ét}}(X)$. An $\mathbb{F}_q$-rational point $x_0 \in X_0(\mathbb{F}_q)$ is fixed by Φ. This gives a pointed morphism $\Phi : (X, x_0) \to (X, x_0)$, which in turn induces a group automorphism:

$$\Phi_* : \pi_1^{\text{ét}}(X, x_0) \to \pi_1^{\text{ét}}(X, x_0)$$

Proposition 5. *Let $F := \Phi_*$.*

1. *The map F is an isomorphism.*
2. *The action of F on $\pi_1^{\acute{e}t}(X)$ coincides (up to a sign) with the conjugation action induced by the generator of $\mathbb{Z}$ in the semidirect product decomposition $W_{X_0} \cong \pi_1^{\acute{e}t}(X) \rtimes \mathbb{Z}$.*

Abelianizing the sequence (17) and taking coinvariants under the action of F leads to the short exact sequence:

$$1 \longrightarrow (\pi_1^{\text{ét}}(X)^{ab})_F \longrightarrow W_{X_0}^{ab} \longrightarrow \mathbb{Z} \longrightarrow 0.$$

Remark 20. The group of coinvariants $(\pi_1^{\text{ét}}(X)^{ab})_F$ is finite. This follows from the Weil conjectures, which imply that for the action of F on $H_1(X, \mathbb{Q}_l)$, the eigenvalue 1 does not occur. This ensures that the cokernel of the map $(F - 1)$ on $H_1(X, \mathbb{Q}_l)$ is trivial, which implies the finiteness of the coinvariants on $H_1(X, \mathbb{Z}_l) \cong \pi_1^{\text{ét}}(X)^{ab} \otimes \mathbb{Z}_l$.

9.3.2 Construction of the Reciprocity Map

We aim to construct the isomorphism $W_{X_0}^{ab} \cong \mathrm{Jac}(X_0)(\mathbb{F}_q) \times \mathbb{Z}$. The map to $\mathbb{Z}$ is simply the canonical projection from the Weil group. The non-trivial part is constructing the map to the Jacobian.

$$W_{X_0}^{ab} \longrightarrow \mathrm{Jac}(X_0)(\mathbb{F}_q)$$

This map arises from the Abel-Jacobi map $AJ_{x_0} : X \to \mathrm{Jac}(X)$, which is functorial and induces a map $W_{X_0} \to W_{\mathrm{Jac}}$ on Weil groups.

The rational points of the Jacobian can be described using the Lang isogeny:

$$\begin{aligned} \mathcal{L} : \mathrm{Jac}(X) &\longrightarrow \mathrm{Jac}(X) \\ y &\longmapsto \Phi(y) - y. \end{aligned}$$

The kernel of this isogeny is precisely the group of rational points, $\ker(L) = \mathrm{Jac}(X)(\mathbb{F}_q) = \mathrm{Jac}(X_0)(\mathbb{F}_q)$, and the Lang isogeny is an étale map. Therefore, it induces a map $W_{\mathrm{Jac}(X)} \to \mathrm{Jac}(X)(\mathbb{F}_q)$. The overall construction can be summarized by the following diagram:

$$\begin{tikzcd} W_{X_0}^{ab} \arrow[r] \arrow[d, two heads] & W_{\mathrm{Jac}(X)}^{ab} \arrow[r, "\mathcal{L}"] & \mathrm{Jac}(X_0)(\mathbb{F}_q) \\ \mathbb{Z}\,. & & \end{tikzcd}$$

10 Lang Isogeny: 7/10/2025

Scribe: Minghan Sun

10.1 Universality of the Lang Isogeny

Theorem 10 (universality of Lang Isogeny)**.** *Suppose A_0, B_0 are commutative connected algebraic groups over $\mathbb{F}_q$. Suppose we are given a short exact sequence*

$$0 \to \Gamma \to B_0 \xrightarrow{\pi} A_0 \to 0 \tag{18}$$

where Γ is a discrete finite group. Then there exists a unique factorization of the above short exact sequence, i.e. a unique map $\alpha : A_0 \to B_0$ such that $\pi \circ \alpha = \mathcal{L}_{A_0}$.

Remark 21. *In other words, theorem 10 says that the universal finite (étale) cover of A_0 by an algebraic group such that the kernel of the cover is discrete is the cover $A_0 \xrightarrow{\mathcal{L}_{A_0}} A_0$.*

Remark 22. *In the statement of theorem 10, by a "discrete" finite group Γ, we mean a finite group Γ with no extra algebraic geometry, i.e. a finite group Γ which is isomorphic to a disjoint union of copies of* $\operatorname{Spec} \mathbb{F}_q$.

Example 23. *Consider the map $f : \mathbb{G}_{m,\mathbb{F}_q} \to \mathbb{G}_{m,\mathbb{F}_q}$ given by $f(t) = t^k$ for some k. When $(k,q) = 1$, f is a finite étale cover of $\mathbb{G}_{m,\mathbb{F}_q}$.*

We have a short exact sequence

$$1 \to \mu_k \to \mathbb{G}_{m,\mathbb{F}_q} \xrightarrow{f} \mathbb{G}_{m,\mathbb{F}_q} \to 1. \tag{19}$$

It is a fact that μ_k is discrete in the sense of remark 22 if and only if $\alpha^\vee \mu_k(\mathbb{F}_q) = k$, which happens if and only if $k = q-1$. When $k = q-1$, μ_k equals $\mathbb{F}_q^\times$.

proof of theorem 10. We have a diagram

$$\begin{tikzcd} 0 \arrow[r] & \Gamma \arrow[r] \arrow[d, hook] & B_0 \arrow[r, "\pi"] \arrow[d, equal] & A_0 \arrow[r] \arrow[d, dotted, "\alpha"] & 0 \\ 0 \arrow[r] & B_0(\mathbb{F}_q) \arrow[r] & B_0 \arrow[r, "\mathcal{L}_{B_0}"] & B_0 \arrow[r] & 0. \end{tikzcd} \tag{20}$$

Here we have an inclusion $\Gamma \hookrightarrow B_0(\mathbb{F}_q)$ because Γ is discrete. Denoted by α the induced map $A_0 \to B_0$. We will show that $\pi \circ \alpha = \mathcal{L}_{A_0}$.

By construction, it is clear that $\alpha\pi = \mathcal{L}_{B_0}$. So $\pi\alpha\pi = \pi\mathcal{L}_{B_0}$. On the other hand, by functoriality, we have $\mathcal{L}_{A_0}\pi = \pi\mathcal{L}_{B_0}$. So $\pi\alpha\pi = \mathcal{L}_{A_0}\pi$. Since π is a surjective map, we have $\pi\alpha = \mathcal{L}_{A_0}$, as desired. $\square$

Definition 14 (Galois cover)**.** *Suppose A_0, B_0 are algebraic groups over $\mathbb{F}_q$ and $p : B_0 \to A_0$ is a cover. We say p is a* Galois cover *if p is finite and étale and the group of Decke transformations of p is acts simply transitively on the fibers of p.*

Corollary 4. *Suppose $A_0/\mathbb{F}_q$ is an abelian variety. Suppose we have a connected Galois cover $p : Y_0 \to A_0$ with the group of Decke transformations equal to Γ. Suppose $y_0 \in Y_0$ such that $p(y_0) = 0$. Then the cover p comes from the Lang Isogeny via the map $A_0(\mathbb{F}_q) \to \Gamma$.*

Proof. We need to show that Y_0 has a group structure which is compatible with the map p.

Let Y, A denote the base changes of Y_0, A_0 to $\overline{\mathbb{F}_q}$, respectively. We know that

$$\{\text{cover } Y \to A\} \leftrightarrow \{\text{a homomorphism } \rho : \pi_1^{ét}(A) \to \Gamma\}. \tag{21}$$

Since we are working over $\overline{\mathbb{F}_q}$, we have the Kunneth Formula. So the RHS of the correspondence above fits into a diagram

$$\begin{tikzcd} \pi_1^{ét}(A) \arrow[rr, "\rho"] & & \Gamma \\ \pi_1^{ét}(A \times A) \arrow[u, "\mathrm{add}_*"] \arrow[r, "\cong"] & \pi_1^{ét}(A) \times \pi_1^{ét}(A) \arrow[ul, "\mathrm{sum}"] \arrow[r] & \Gamma \times \Gamma. \arrow[ul, "\mathrm{sum}"] \end{tikzcd} \tag{22}$$

Unwinding what the diagram means on the LHS of the above correspondence, we see that Y has a group structure compatible with the map $Y \to A$.

To finish the proof, we notice that we can define the desired group structure on Y_0 (compatible with p) by Galois descent. $\square$

Remark 23. *We already know that $A_0 \xrightarrow{\mathcal{L}_{A_0}} A_0$ is a Galois cover of A_0 with group of Decke transformations equal to $A_0(\mathbb{F}_q)$. corollary 4 says that this cover of A_0 is the universal example of a Galois cover of A_0 with a finite discrete group of Decke transformations.*

Suppose $Z_0/\mathbb{F}_q$ is geometrically connected and let $Z = Z_0|_{\overline{\mathbb{F}_q}}$. Recall from the previous lecture that we have a short exact sequence

$$1 \to \pi_1^{ét}(Z) \to \pi_1^{ét}(Z_0) \to \pi_1^{ét}(\mathbb{F}_q) = \hat{\mathbb{Z}} \to 0. \tag{23}$$

If we choose a point $z_0 \in Z_0(\mathbb{F}_q)$, then we get a section $\pi_1^{ét}(\mathbb{F}_q) = \hat{\mathbb{Z}} \to \pi_1^{ét}(Z_0)$. As a result, we have

$$\pi_1^{ét}(Z_0) = \hat{\mathbb{Z}} \ltimes \pi_1^{ét}(Z). \tag{24}$$

This gives us an automorphism F of $\pi_1^{ét}(Z)$ and we let $\pi_1^{ét}(Z)_F$ denote its group of covariants.

Theorem 11. *Notation as above. Then the data of a Γ-cover $p : Y_0 \to Z_0$ (Γ discrete and finite) plus the data of a point $y_0 \in Y_0(\mathbb{F}_q)$ with $p(y_0) = z_0$ is equivalent to the data of a homomorphism $\pi_1^{ét}(Z)_F \to \Gamma$.*

Proof. We know that

$$\left\{\begin{matrix} \text{A } \Gamma\text{-cover } p : Y_0 \to Z_0 \\ \text{plus a point } y_0 \in Y_0(\overline{\mathbb{F}_q}) \text{ lying over } z_0 \end{matrix}\right\} \leftrightarrow \{\text{a homomorphism } \pi_1^{ét}(Z_0, z_0) \to \Gamma\}. \tag{25}$$

Since we know that $\pi_1^{ét}(Z_0, z_0) = \hat{\mathbb{Z}} \ltimes \pi_1^{ét}(Z)$, these are also equivalent to the data of a homomorphism $\hat{\mathbb{Z}} \ltimes \pi_1^{ét}(Z) \to \Gamma$.

If we require the point $y_0 \in Y_0(\overline{\mathbb{F}_q})$ to be $\mathbb{F}_q$-rational, we get a correspondence

$$\left\{\begin{matrix} \text{A } \Gamma\text{-cover } p : Y_0 \to Z_0 \\ \text{plus a point } y_0 \in Y_0(\mathbb{F}_q) \text{ lying over } z_0 \end{matrix}\right\} \leftrightarrow \left\{\begin{matrix} \text{a homomorphism } \hat{\mathbb{Z}} \ltimes \pi_1^{ét}(Z) \to \Gamma \\ \text{trivial on } \hat{\mathbb{Z}} \end{matrix}\right\}. \tag{26}$$

But we also know that

$$\left\{\begin{matrix} \text{a homomorphism } \hat{\mathbb{Z}} \ltimes \pi_1^{ét}(Z) \to \Gamma \\ \text{trivial on } \hat{\mathbb{Z}} \end{matrix}\right\} \leftrightarrow \{\text{a homomorphism } \pi_1^{ét}(Z)_F \to \Gamma\}, \tag{27}$$

finishing the proof. $\square$

Corollary 5. *Suppose A_0 is an abelian variety over $\mathbb{F}_q$. Then*

$$\pi_1^{ét}(A)_F = A_0(\mathbb{F}_q). \tag{28}$$

Proof. Corollary of previous results. $\square$

10.2 Application to CFT

Recall our setting from the last lecture. We let X_0 be a smooth, projective, and geometrically connected algebraic curve over $\mathbb{F}_q$ and $X = X_0|_{\overline{\mathbb{F}_q}}$. We have a short exact sequence

$$1 \to \pi_1^{ét}(X) \to W_{X_0} \to \mathbb{Z} \to 0. \tag{29}$$

Choosing some $x_0 \in X_0(\mathbb{F}_q)$, we get a splitting $\mathbb{Z} \to W_{X_0}$ of the above short exact sequence. As a result, we can write

$$W_{X_0} = \mathbb{Z} \ltimes \pi_1^{ét}(X), \tag{30}$$

and so we get an automorphism F of $\pi_1^{ét}(X)$.

Abelianize the short exact sequence above and taking F-coinvariants, we get a short exact sequence

$$1 \to \pi_1^{ét}(X)_F^{\mathrm{ab}} \to W_{X_0}^{\mathrm{ab}} \to \mathbb{Z} \to 0 \tag{31}$$

along with a section $\mathbb{Z} \to W_{X_0}^{\mathrm{ab}}$. As a result, we have

$$W_{X_0}^{\mathrm{ab}} = \mathbb{Z} \times \pi_1^{ét}(X)_F^{\mathrm{ab}}. \tag{32}$$

Also, from Geometric CFT, we have

$$\pi_1^{ét}(X)^{\mathrm{ab}} = \pi_1^{ét}(\mathrm{Jac}(X)). \tag{33}$$

Since $\mathrm{Jac}(X)$ is an abelian variety, by corollary 5, we have

$$\pi_1^{ét}(\mathrm{Jac}(X))_F = \mathrm{Jac}(X)(\mathbb{F}_q). \tag{34}$$

To summarize, we have shown that

$$W_{X_0} = \mathbb{Z} \times \mathrm{Jac}(X)(\mathbb{F}_q), \tag{35}$$

which is what we wanted to show.

11 Modular forms: 09/10/2025

Scribe: Zachary Carlini

11.1 Motivation from Modular Forms

For a more detailed treatment of modular forms, see [Ser73], [DS05], etc...

"Every human being should read Serre's *A Course in Arithmetic*" - Sam Raskin

The theory of modular forms is not strictly neccessary for us, but it will provide some helpful intuition and motivation.

By a class vote, we will start with the more concrete definition of modular forms, so we will view them as certain functions on the upper half plane $\mathbb{H} = \{\tau \in \mathbb{C} : \Im(\tau) > 0\}$.

Definition 15 (Holomorphic Modular Forms of Level 1, no Nebentypus)**.** *A modular form of weight k is a holomorphic function $f : \mathbb{H} \to \mathbb{C}$ satisfying the following properties:*

- *(Modularity) For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ in $\mathrm{SL}_2(\mathbb{Z})$ and $\tau \in \mathbb{H}$,*

$$f\left(\frac{a\tau + b}{c\tau + d}\right) = (c\tau + d)^k f(\tau).$$

- *(Growth Condition) For every $\lambda > 0$, f is bounded on the region $\{\tau \in \mathbb{H} : \mathrm{im}(\tau) > \lambda\}$.*

Remark 24. *If f is a modular form of weight k, we must have $f(\tau) = f\left(\frac{-\tau+0}{0+-1}\right) = (-1)^k f(\tau)$, so for f to be nonzero, k must be even.*

Remark 25. *If f is any modular form, then $f\left(\frac{\tau+1}{0\tau+1}\right) = f(\tau)$, so f is periodic. The quotient of $\mathbb{H}$ by the $\mathbb{Z}$ action determined by $\tau \mapsto \tau + 1$ is $\mathbb{D}^\circ = \{z \in \mathbb{C} : 0 < \|z\| < 1\}$, where the quotient map is realized by the exponential map $\mathbb{H} \xrightarrow{\tau \mapsto e^{2\pi i \tau}} \mathbb{D}^\circ$. Therefore, any modular form factors through a function $\mathbb{D}^\circ \to \mathbb{C}$, which by abuse of notation we will also denote by f. To distinguish these two functions, we will always denote the coordinate on $\mathbb{D}^\circ$ by q, whereas we will denote the coordinate on $\mathbb{H}$ by τ, so we will write: $f(\tau) = f(q)$, where it is understood that $\tau \in \mathbb{H}$ and $q = e^{2\pi i \tau} \in \mathbb{D}^\circ$.*

The growth condition in the definition of modular forms implies that any modular form $f(q)$ extends to a holomorphic function on $\mathbb{D} = \{z \in \mathbb{C} : \|z\| < 1\}$, so we may write:

$$f(q) = \sum_{n \geq 0} a_n q^n$$

for some coefficients $a_n \in \mathbb{C}$. This is called the *Fourier expansion* or the *q-expansion* of the modular form f.

Remark 26. *If f is a modular form of weight k, then $f\left(\frac{1}{\tau}\right) = \tau^k f(\tau)$. In terms of q-expansions, this means:*

$$\tau^k \sum_{n=0}^{\infty} a_n e^{2\pi i n \tau} = \sum_{n=0}^{\infty} a_n e^{-\frac{2\pi i n}{\tau}}$$

which is ugly and complicated looking. The takeaway is that modularity is not easy to check from the perspective of q-expansions.

Example 24 (Eisenstein Series)**.** *For $k \geq 2$,*

$$G_{2k}(\tau) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus 0} \frac{1}{(m\tau + n)^{2k}}$$

is a weight $2k$ modular form. This is called the Eisenstein series *of weight $2k$.*

Example 25 (Ramanujan Δ Function, a.k.a Ramnujan Discriminant Function)**.**

$$\Delta(q) = q\prod(1-q^n)^{24} = \eta(\tau)^{24}$$

is modular of weight 12*. What does this function do? There is a simpler identity:*

$$\prod_n (1+q^n) = \sum_n a_n q^n,$$

where a_n is the number of ways to write n as a sum of distinct positive integers. If we change our generating function to $\prod_n (1+q^n)^{24}$, then the coefficients in the expansion count the number of ways to write n as a sum of positive integers, where we allow each positive integer to be repeated at most 24 *times, and we weight each expression by a combinatorial term that counts the number of ways we could have selected* those *copies of the summands out of the* 24 *available. Finally, the coefficients in the expansion of our actual $\Delta(q)$, which is given by $q\prod_n (1-q^n)^{24}$, count the (weighted) number of ways to write $n-1$ as a sum of positive integers, where we allow a summand to be repeated at most* 24 *times, and we attach a sign to the weights which depends on the multiplicity of each summand.*

We will denote by $\mathscr{M}_k$ the set of holomorphic modular forms of weight k, and we will denote by $\mathscr{S}_k$ the set of *cuspidal* modular forms of weight k. A modular form $f(q) = \sum_{n=0}^{\infty} a_n q^n$ is called cuspidal if $a_0 = 0$, or, equivalently, if $f(q)$ vanishes at $q = 0$ (the cusp).

Here are some lovely, very special facts that are specific to the holomorphic level 1 case:

- $\bigoplus_{k\in\mathbb{Z}} M_k \cong \mathbb{C}[G_4, G_6]$ as a graded ring, where G_4 has degree 4 and G_6 has degree 6.
- $\dim \mathscr{S}_k = \dim \mathscr{M}_k - 1$ whenever $\mathscr{M}_k$ is nonzero. In particular, $\dim \mathscr{S}_{12} = 1$, so it is spanned by Δ.

The weight k encodes something about the Archimedean place, so when we go to function fields, it won't appear.

We will now introduce a nicer, more abstract definition of modular forms.

Definition 16 (Abstract)**.** *A modular form of weight $2k$ is an assignment of $\Lambda \subset L \mapsto f(\Lambda, L) \in L^{\otimes -k}$ for L a complex line and $\Lambda \subset L$ a lattice such that f is holomorphic and bounded in a suitable sense (e.g. the sense that makes this definition equivalent to the concrete one).*

Dictionary: If f is an abstract modular form of weight k, then given $\tau \in \mathbb{H}$, we can plug $L = \mathbb{C}$ and $\Lambda = \mathbb{Z} \oplus \tau\mathbb{Z}$ into f to get a number, so this defines a function $\mathbb{H} \to \mathbb{C}$ which will be a concrete modular form of weight k. Conversely, if f is a concrete modular form of weight k, then given $\Lambda \subset L$, choose a basis $\Lambda = \operatorname{span}_{\mathbb{Z}}(\omega_1, \omega_2)$ with $\omega_1/\omega_2 \in \mathbb{H}$ (note that for any basis ω_1, ω_2, exactly one of ω_1/ω_2 and ω_2/ω_1 is in $\mathbb{H}$, so we can choose a basis ω_1, ω_2 arbitrarily and then swap them if we need to in order to get $\omega_1/\omega_2 \in \mathbb{H}$). Then we can use ω_2 as a basis for L, so there is a unique linear functional $\alpha_{\omega_1,\omega_2} : L^{\otimes k} \to \mathbb{C}$ characterized by $\alpha_{\omega_1,\omega_2}\left(t.\omega_2^{\otimes k}\right) = tf(\omega_1/\omega_2)$ for $t \in \mathbb{C}$. If we had chosen η_1, η_2 instead of ω_1, ω_2 as our basis, we could similarly produced a linear functional $\alpha_{\eta_1,\eta_2} : L^{\otimes -k} \to \mathbb{C}$ characterized by $\alpha_{\eta_1,\eta_2}\left(t.\eta_2^{\otimes k}\right) = tf(\eta_1/\eta_2)$. There is a unique matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ with $a\omega_1 + b\omega_2 = \eta_1$ and $c\omega_1 + b\omega_2 = \eta_2$, and by the modularity of f,

$$f\left(\frac{\eta_2}{\eta_1}\right) = f\left(\frac{a\omega_1 + b\omega_2}{c\omega_1 + d\omega_2}\right) = (c(\omega_1/\omega_2) + d)^k f(\omega_1/\omega_2) = \left(\frac{\eta_2}{\omega_2}\right)^k f(\omega_1/\omega_2).$$

In particular, for $t \in \mathbb{C}$, we have:

$$\alpha_{\eta_1,\eta_2}\left(t\eta_2^{\otimes k}\right) = tf(\eta_1/\eta_2) = t\left(\frac{\eta_2}{\omega_2}\right)^k f(\omega_1/\omega_2) = \left(\frac{\eta_2}{\omega_2}\right)^k \alpha_{\omega_1,\omega_2}\left(t\omega_2^{\otimes k}\right) = \alpha_{\omega_1,\omega_2}\left(t\eta_2^{\otimes k}\right),$$

so $\alpha_{\eta_1,\eta_2} = \alpha_{\omega_1,\omega_2}$. Therefore, $\alpha_{\omega_1,\omega_2}$ does not actually depend on the choice of ω_1, ω_2, so this procedure gives us a canonical assignment $\Lambda \subset L \mapsto \Lambda^{\otimes -k}$, and this is an abstract modular form.

Remark 27. *A complex line with a lattice corresponds to an elliptic curve over $\mathbb{C}$. Given $\Lambda \subset L$, we associate the elliptic curve L/Λ with basepoint the projection of $0 \in L$. Conversely, given an elliptic curve E with basepoint 0, we can take $L = T_0E$ and we can take Λ to be the kernel of the exponential map from T_0E to E.*

Here is a group theoretic perspective on the coincidence of the abstract and concrete definitions of modular forms: since $\mathrm{SL}_2(\mathbb{R})$ acts transitively on $\mathbb{H}$, and the stabilizer of a point is conjugate to $\mathrm{SO}_2(\mathbb{R})$ (note that $\mathrm{SO}_2(\mathbb{R}) = S^1 \subset \mathbb{C}^\times$), we can write $\mathbb{H} \cong \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Then we have $\mathrm{SL}_2(\mathbb{Z})\backslash H \cong \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$.

There is also a transitive action of $\mathrm{GL}_2(\mathbb{R})$ on the space of lattices for a given complex line L, and the stabilizer of a point under this action is conjugate to $\mathrm{GL}_2(\mathbb{Z})$. The automorphism group of the line L is $\mathbb{C}^\times$, so a function on pairs $\Lambda \subset L$ which is invariant under simultaneous isomorphism is a function on $\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times$. But

$$\mathrm{GL}_2(\mathbb{Z})\backslash \mathrm{GL}_2(\mathbb{R})/\mathbb{C}^\times \cong \mathrm{SL}_2(\mathbb{Z})\backslash \left(\{\pm 1\}\backslash \mathrm{GL}_2(\mathbb{R})/\mathbb{R}^\times_{>0}\right)/\mathrm{SO}_2(\mathbb{R}) \cong \mathrm{SL}_2(\mathbb{Z})\backslash \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R}),$$

which gives the equivalence of the two definitions of modular forms.

Recall that for $\tau \in \mathbb{H}$, we defined the Eisenstein series $G_{2k}(\tau)$ as the concrete modular form given by the formula:

$$G_{2k}(\tau) = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{1}{(m\tau+n)^{2k}} \tag{36}$$

I claim that under our dictionary, this corresponds to the abstract modular form given by:

$$G_{2k}(\Lambda \subset l) = \sum_{\lambda\in\Lambda\setminus 0} \lambda^{\otimes -2k} \in L^{\otimes -2k}. \tag{37}$$

To check this, we need to plug $\mathbb{Z} \oplus \tau\,\mathbb{Z} \subset \mathbb{C}$ into our abstract modular form and compute:

$$G_{2k}(\mathbb{Z}\oplus\tau\,\mathbb{Z} \subset \mathbb{C}) = \sum_{\lambda\in(\mathbb{Z}\oplus\tau\,\mathbb{Z})\setminus\{0\}} \lambda^{-2k} = \sum_{(m,n)\in\mathbb{Z}^2\setminus 0} \frac{1}{(m\tau+n)^{2k}}.$$

Note that equation (37) immediately implies that $G_{2k}(\tau)$, viewed as a function on $\mathbb{H}$, transforms correctly under the $\mathrm{SL}_2(\mathbb{Z})$ action, while equation (36) immediately implies that $G_{2k}(\tau)$ is holomorphic and satisfies the growth condition.

12 Hecke operators: 14/10/2025

Scribe: Joakim Færgeman

12.1 Hecke Theory

12.1.1

We have seen that modular forms can be regarded as functions

$$f : (\Lambda, L) \mapsto f(\Lambda, L) \in L^{-k},$$

where L is a complex vector space of dimension 1, and $\Lambda \subset L$ is a lattice.

For each integer $n \geq 1$, we introduce the operator

$$T_n : \mathcal{M}_k \to \mathcal{M}_k, \ (T_n f)(\Lambda, L) = n^{k-1} \sum_{[\Lambda:\Lambda']=n} f(\Lambda', L).$$

Here, the sum runs over all subgroups of Λ of index n (which are automatically lattices). We call T_n the n'th Hecke operator.

12.1.2

By direct inspection, we see that:

1. $T_1 = \mathrm{id}$.
2. $T_n \circ T_m = T_{nm}$ whenever $\gcd(n, m) = 1$.
3. For any prime number p, we have

$$T_{p^n} \circ T_p = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}.$$

Remark 28. *The third identity is analogous to the following. Let V be a vector space of dimension two over some field. Then:*

$$(\mathrm{Sym}^n V) \otimes V = \mathrm{Sym}^{n+1} V \oplus \mathrm{Sym}^{n-1} V \otimes \Lambda^2 V.$$

Remark 29. *The Hecke operators preserve the subspace $\mathcal{S}_k$ of cusp forms.*

A consequence of the above three identities is the following:

Corollary 6. *The T_n's commute.*

Remark 30. *The T_n's are simultaneously diagonalizable. Indeed, by Corollary 6, it suffices to show that each T_n is diagonalizable. This in turn follows from the fact that we have a Hermitian bilinear pairing called the Petersson inner product:*[6]

$$\langle \cdot, \cdot \rangle : \mathcal{M}_k \times \mathcal{S}_k \to \mathbb{C}$$

which is non-degenerate when restricted to $\mathcal{S}_k \times \mathcal{S}_k$, and that T_n is self-adjoint with respect to this pairing.

Definition 17. *A* Hecke eigenform *is an eigenvector $f \in \mathcal{M}_k$ for the T_n's. Note that it is sufficient to be an eigenvector for T_p for each prime p.*

[6] We will not define this inner product here, but the reader should feel free to look it up.

Example 26. *Recall the Eisenstein series of even weight $k \geq 4$, written as a function on (Λ, L):*

$$G_k(\Lambda, L) = \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k} \in L^{-k}.$$

We claim that G_k is a Hecke eigenform. We check this directly. Let p be a prime number. Then:

$$(T_p G_k)(\Lambda, L) = p^{k-1} \sum_{[\Lambda:\Lambda']=p} G_k(\Lambda', L) = p^{k-1} \sum_{[\Lambda:\Lambda']=p} \sum_{0 \neq \lambda \in \Lambda'} \lambda^{-k}$$

We distinguish two cases:

- *If $\lambda \in p\Lambda$, then λ is contained in any subgroup Λ' of Λ of index p, since $p\Lambda \subset \Lambda'$. In this case, there are exactly $|\mathbb{P}^1(\mathbb{F}_p)| = p+1$ subgroups of $\Lambda/p\Lambda \simeq (\mathbb{Z}/p\mathbb{Z})^2$ of index p.*
- *If $\lambda \notin p\Lambda$, then there is a unique Λ' of index p containing λ. This is saying that there is a unique line in $\Lambda/p\Lambda \simeq (\mathbb{Z}/p\mathbb{Z})^2$ containing the non-zero vector λ.*

As such the above double sum becomes:

$$p^{k-1}\big(\sum_{0 \neq \lambda \in p\Lambda} (p+1)\lambda^{-k} + \sum_{\lambda \notin p\Lambda} \lambda^{-k}\big) = p^{k-1}\big(\sum_{0 \neq \lambda \in p\Lambda} p\lambda^{-k} + \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}\big)$$

$$p^{k-1}\big(\sum_{0 \neq \lambda \in \Lambda} p^{1-k}\lambda^{-k} + \sum_{0 \neq \lambda \in \Lambda} \lambda^{-k}\big) = (1 + p^{k-1}) \cdot G_k(\Lambda, L).$$

Example 27. *From here, one can check that for all $n \geq 1$:*

$$T_n G_k = \sigma_{k-1}(n) G_k,$$

where $\sigma_k(n) = \sum_{d|n} d^k$.

12.2 Relationship to Fourier Coefficients

12.2.1

For the standard lattice $\mathbb{Z}^2$, we write $e_1 = (1,0), e_2 = (0,1) \in \mathbb{Z}^2$.

Lemma 3. *Let $\Lambda \subset \mathbb{Z}^2$ be an index n subgroup. There exist unique integers (c,d) with $d \geq 1, d|n, 0 \leq c < d$ such that*

$$\Lambda = \mathrm{Span}_{\mathbb{Z}}(de_1, ce_1 + \frac{n}{d}e_2).$$

Proof. Since $ne_1 \in \Lambda$, choose $d|n$ minimal such that $de_1 \in \Lambda$. Next, consider an element $\alpha e_1 + \beta e_2 \in \Lambda$, where $x = \alpha, \beta \in \mathbb{Z}$ and $\beta \geq 1$ is minimal. By adding multiples of de_1 to x, we may assume that $c := \alpha$ satisfies that $0 \leq c < d$.

We claim that $\beta = \frac{n}{d}$, which finishes the proof.

□

12.2.2

Recall that given τ in the upper half plane $\mathbb{H}$, we may associate the lattice $\mathbb{Z} \oplus \mathbb{Z} \cdot \tau \subset \mathbb{C}$. Given $d|n$, and $0 \leq c < d$, we get the sublattice $\mathbb{Z} \cdot d \oplus \mathbb{Z}(\frac{n}{d}\tau + c) \subset \mathbb{Z} \oplus \mathbb{Z} \cdot \tau$ of index n.

Multiplication by $d : \mathbb{C} \to \mathbb{C}$ sends this lattice to $\mathbb{Z} \oplus \mathbb{Z} \cdot (\frac{n}{d^2}\tau + \frac{c}{d})$. Now we can write what the Hecke operators do to modular forms when we consider the latter as functions on the upper half plane. Namely, we get the formula:

$$T_n f(\tau) = n^{k-1} \sum_{d|n, 0 \leq c < d} f(\frac{n}{d^2}\tau + \frac{c}{d}) d^{-k}.$$

12.2.3

Next, we turn to the question of what Hecke operators do to Fourier coefficients. Recall that a modular form (of level 1) has a $q = e^{2\pi i\tau}$-expansion:

$$f(q) = \sum_{n\geq 0} a_n(f)q^n.$$

Note that:

$$T_nf(\tau) = n^{k-1}\sum_{m\geq 0}\sum_{d|n, 0\leq c<d} a_m(f)e^{2\pi i m(\frac{n}{d^2}\tau+\frac{c}{d})}d^{-k}$$

$$= n^{k-1}\sum_{m\geq 0}\sum_{d|n} a_m(f)\cdot e^{2\pi i m\frac{n}{d^2}\tau}d^{-k}\sum_{0\leq c<d} e^{2\pi i m\frac{c}{d}}$$

Note that the sum $\sum\limits_{0\leq c<d} e^{2\pi i m\frac{c}{d}}$ equals zero unless $d|m$, in which case it equals d. Replacing m by dm, we get:

$$T_nf(\tau) = n^{k-1}\sum_{m\geq 0} a_{dm}(f)\sum_{d|n} e^{2\pi i m\frac{n}{d}\tau}d^{1-k}$$

$$= n^{k-1}\sum_{m\geq 0}\Big(\sum_{d|n} d^{1-k}a_{dm}(f)\Big)e^{2\pi i m\frac{n}{d}\tau}$$

Example 28. *Note that:*

$$a_0(T_nf) = n^{k-1}\sum_{d|n} d^{1-k}a_0(f) = a_0(f)\cdot\sigma_{k-1}(n).$$

Example 29.

$$a_1(T_nf) = n^{k-1}\sum_{m\geq 0, d|n, mn=d} d^{1-k}a_{dm}(f)$$

$$= n^{k-1}\cdot n^{1-k}\cdot a_n(f) = a_n(f).$$

In the theory of automorphic forms, this is known as the 'Casselman-Shalika formula'.

12.2.4

Recall that Hecke operators preserve cusp forms. Suppose know that $f = \sum\limits_{n\geq 1} a_n(f)q^n$ is a cuspidal Hecke eigenform. Then

$$T_nf = \lambda_nf = \sum_{n\geq 1}\lambda_na_n(f)q^n$$

for some $\lambda_n \in \mathbb{C}$. By Example 29, we have

$$a_n(f) = \lambda_na_1(f).$$

This also implies that $a_1(f) \neq 0$. Without loss of generality, we can rescale f to assume that $a_1(f) = 1$ (in which case we refer to f as a 'normalized' eigenform). In this case

$$a_n(f) = \lambda_n.$$

That is, the eigenvalue for T_n is the n'th Fourier coefficient.

12.2.5

Let us apply this to the Ramanujan tau function. Recall that:

$$\Delta(q) = q \cdot \prod_{n \geq 1} (1 - q^n)^{24} = \sum_{n \geq 1} \tau(n) q^n$$

is the unique normalized cuspidal Hecke eigenform of weight 12. From the discussion in §12.2.4 and the properties of Hecke operators listed in §12.1.2, we get:

1. $\tau(nm) = \tau(n)\tau(m)$ whenever $\gcd(n, m) = 1$.
2. $\tau(p^n)\tau(p) = \tau(p^{n+1}) + p^{11}\tau(p^{n-1})$ for any prime p.

13 What did Langlands Predict: 23/10/2025

Scribe: Michael Horzepa

Let us recall that what we have so far is the space of weight k holomorphic (cuspidal) modular forms $\mathcal{M}_k(\supseteq \mathcal{S}_k)$ as well as the Hecke operators T_n which act on them. The T_n's are simultaneously diagonalizable, preserve the space of cusp forms, and each $T_n \in \mathbb{C}[\{T_p\}]_{p \text{ prime}}$. We now want to expand upon what the Langlands program actually predicts for these types of functions.

13.1 What do the Langlands conjectures say?

When trying to answer as to what the Langlands conjectures actually say, the answer will by nature be a bit mysterious. When Langlands first started thinking about the ideas, they were just that: ideas and a philosophy. But, in the USSR one could not speak of philosophy, so when Drinfeld gave his PhD dissertation, he used the word conjectures and it has stuck ever since. Because of that, we will give only a flavor of the answer in this particular set up.

Taking the Langlands philosophy as a sort of "religious belief", there is supposed to be a topological group, and in fact a pro-Lie group, $L_{\mathbb{Q}}$ of Weil group flavor. The regular Weil group of $\mathbb{Q}$ is in fact too small which mandates the existence of the Langlands group of $\mathbb{Q}$. In the function field case $F = \mathbb{F}_q(X_0)$, this group takes on the role of $\mathbb{W}_F$ for the most part. Then there should be an isomorphism

$$L_{\mathbb{Q}}^{\wedge} \simeq Gal_{\mathbb{Q}}$$

Where the left hand side is the profinite completion. There should also exist a map $|\cdot| : L_{\mathbb{Q}} \to \mathbb{R}^{>0}$ such that for each prime p there should be a well defined up to conjugation Weil-Deligne map from $\mathbb{Q}_p$ to $L_{\mathbb{Q}}$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{W}_{\mathbb{Q}_p} \times \text{something} \arrow[r, hook] \arrow[d] & L_{\mathbb{Q}} \arrow[d, "|\cdot|"] \\
\mathbb{Z} \arrow[r, "n \mapsto p^n"] & \mathbb{R}^{>0}
\end{tikzcd}$$

There is also a map from the Weil group of $\mathcal{W}_{\mathbb{R}} \hookrightarrow L_{\mathbb{Q}}$, but we haven't discussed this. The key takeaway is that this offers some extension of Galois theory which is not purely algebraic, but rather also incorporates an analytic flavor. Of course, class field theory should say there exists an isomorphism:

$$L_{\mathbb{Q}}^{ab} \simeq \mathbb{A}_{\mathbb{Q}}^{\times}/\mathbb{Q}^{\times}$$

There is also an everywhere unramified version which looks like the following for each prime p:

$$\begin{tikzcd}
L_{\mathbb{Z}} \arrow[r, phantom, "\supseteq"] & \mathbb{Z} = \{Fr_p^n\} \\
L_{\mathbb{Q}} \arrow[u]
\end{tikzcd}$$

This $L_{\mathbb{Z}}$ will also contain $\mathcal{W}_{\mathbb{R}}$.

13.2 The Vague Picture

One should have the following vague picture in mind:

$$\begin{tikzcd}
\operatorname{Spec} \mathbb{F}_p \arrow[r, hook] & \operatorname{Spec} \mathbb{Z} \arrow[d] \\
& \operatorname{Spec} \mathbb{F}_1
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{Z} \arrow[d, equal] \arrow[r, phantom, "1 \mapsto Fr_p"] & L_{\mathbb{Z}} \arrow[d, equal] \arrow[dd, bend left, "|\cdot|"] \\
\pi_1(\operatorname{Spec} \mathbb{F}_p) \arrow[r] \arrow[dr, "1 \mapsto p"] & \pi_1(\operatorname{Spec} \mathbb{Z}) \arrow[d] \\
& \pi_1(\operatorname{Spec} \mathbb{F}1) = \mathbb{R}^{>0}
\end{tikzcd}$$

with the right side being induced by the left. Here $\mathbb{F}_1$ denotes the field with one element. The principle here is described by:

$$\begin{tikzcd} \{\text{motives}/\mathbb{Q}\} \arrow[r] & \text{rep'ns of } L_{\mathbb{Q}} \text{ which look alg.} \\ \{\text{Varieties}/\mathbb{Q}\} \arrow[u, "H^\bullet_{\text{univ}}"] & \end{tikzcd}$$

The idea is that every reasonable cohomology theory on varieties should factor through motives, and that sufficiently nice "Galois" representations should come from motives. As an example, consider the following which comes from the Tate motive $\mathbb{Q}(-1)$:

$$\begin{tikzcd} \mathbb{Q}(-1) \arrow[r] \arrow[d, equal] & |\cdot|^{(H)} : L_{\mathbb{Q}} \to \mathbb{R}^{>0} \\ H^2(\mathbb{P}^1_{\mathbb{Q}}) & \end{tikzcd}$$

We could also have a different map such as $|\cdot|^{\sqrt{2}}$, but this doesn't come from a motive.

Note that the idea in some sense is:

$$\{\text{motives}/\mathbb{Q}\} \xrightarrow{H^\bullet_{\acute{e}t}(-,\mathbb{Q}_\ell)} \left\{ \begin{array}{c} \mathbb{Q}_\ell\text{-vector spaces w/ a} \\ \text{cts. action of } \mathrm{Gal}_{\mathbb{Q}} \end{array} \right\}$$

If we stick with the $\mathbb{Q}_\ell$ coefficients then really this is generated by $\mathrm{Gal}_{\mathbb{Q}}$, but if the coefficients are in $\mathbb{C}$, then we instead use just $L_{\mathbb{Q}}$. Then very roughtly Langlands predicts a correspondence:

$$\left\{ \begin{array}{c} \text{automorphic forms} \\ \text{for } \mathcal{G}/\mathbb{Q} \end{array} \right\} \leftrightarrow \left\{ \begin{array}{c} L_{\mathbb{Q}} \to \check{\mathcal{G}}(\mathbb{C}) \\ \text{representations} \end{array} \right\}$$

Here $\check{\mathcal{G}}$ denotes the Langlands dual group which comes from swapping roots and dual roots. In this picture the hope is that cuspidal forms should correspond to irreducible representations.

13.3 Basic Expectations

We have not yet discussed what exactly automorphic forms are, but we claim the modular forms we have seen are actually automorphic forms for $\mathcal{G} = \mathrm{GL}(2) = \check{\mathcal{G}}$. What's really happening is a correspondence:

$$\{f \text{ an eigenform of wt k}\} \leftrightarrow \left\{ \begin{array}{c} \rho_f : L_{\mathbb{Q}} \to \mathrm{GL}_2(\mathbb{C}) \\ \det \rho_f = |\cdot|^{k-1} \end{array} \right\}$$

So in the case of modular forms, we first recall that for a normalized eigenform $T_p f = \lambda_p f$ for $\lambda \in \mathbb{C}$. We expect that $\lambda_p = \mathrm{tr}(\rho_f(Fr_p))$. Let us consider this for the explicit eigenform examples of the Eisenstein series G_{2k}. Then these correspond to the representation

$$\rho_{G_{2k}}(g) = \begin{pmatrix} 1 & 0 \\ 0 & |g|^{2k-1} \end{pmatrix}$$

Then if we take the trace at $\rho_{G_{2k}}(p) = \begin{pmatrix} 1 & 0 \\ 0 & p^{2k-1} \end{pmatrix}$, this exactly aligns with the eigenvalues for T_p we calculated earlier.

Now imagine instead we started with a suitable irreducible 2-dimensional representation ρ of $L_{\mathbb{Q}}$ (really this should come from a motive IRL[7]). Then the belief is that there exists a unique cuspidal eigenform of weight k that comes from this correspondence. Let's try to reconstruct it via the q-expansion $f = a_0 + a_1 q + a_2 q^2 + \ldots$.

For starters, by assumption $a_0 = 0$ and $a_1 = 1$. The for next coefficient, we should have $a_2 = \mathrm{tr}(\rho(Fr_2))$, and similarly for all other primes $a_p = \mathrm{tr}(\rho(Fr_p))$. We saw before that $T_{p^n} T_p = T_{p^{n+1}} + p^{k-1} T_{p^{n-1}}$ (like $(\mathrm{Sym}^n V) \otimes V = \mathrm{Sym}^{n+1} V \oplus (\mathrm{Sym}^{n-1} V) \otimes \wedge^2 V$ for V 2 dimensional). Then this implies that for $n \geq 1$ that

[7] in real life

$$a_{p^{n+1}} = a_{p^n} \cdot a_p - p^{k-1} a_{p^{n-1}}$$

Finally if we recall that for $(n, m) = 1$ we get $a_{mn} = a_m a_n$, then we actually have generated our entire q-expansion.

Of course, this is all well and good to get an infinite series, but at no point in this process did we show that the function we generated was actually modular. This shines light on what Taylor, Wiles, etc. did. They actually went in and checked that the function f generated in this fashion was modular for ρ coming from an (at least a certain class of) elliptic curves.

14 Functions to Sheaves: 28/10/2025

Scribe: Soumik Ghosh

Aim: Geometrization

Recall the analogy for F a global field.

$$\mathbb{R} \supset \mathbb{Z} \leftrightarrow \mathbb{A}_F \supset F$$

Modular forms are functions on $SL_2\mathbb{Z}\backslash\mathbb{H} = SL_2\mathbb{Z}\backslash SL_2\mathbb{R}/SO_2\mathbb{R}$.

Let G be a reductive group (split/ defined over F). In char 0, this is the same as connected affine algebraic groups with semi-simple representation theory. Examples are $GL_n, SL_n, PGL_n, Sp_{2n}, SO_n, Spin_n, G_2, F_4, E_6, E_7, E_8$.

We look at adelic points $G(\mathbb{A})$ and consider $G(F)\backslash G(\mathbb{A})/K$ where $K \subset \mathbb{A} = \prod'_v F_v$ is the subgroup $K = \prod K_v$ where $K_v \subset G(F_v)$, for almost all v, $K_v = G(\mathcal{O}_v)$, for finite v, $K_v \subset G(\mathcal{O}_v)$ is open subgroup and for v Archiemdian, $K_v =$ maximal compact in $G(F_v)$.

Morally: An automorphic form for G is a function "of level K"

$$G(F)\backslash G(\mathbb{A})/K \to \mathbb{C}$$

satisfying certain conditions (too complicated).

Fact: Modular forms of weight $k \leftrightarrow$ automorphic forms for PGL_2 for $F = \mathbb{Q}$.

Let $X_0/\mathbb{F}_q$ be a curve and $F = \mathbb{F}_q(X_0)$. Set $K = \prod_{x\in X_0 \text{ closed}} G(\mathcal{O}_x)$

Say X/k is a smooth projective variety.

Definition 18. $\mathrm{Bun}_G(X)$ *is the stack defined by*

$$\mathrm{Hom}(S, \mathrm{Bun}_G(X)) = \{G\text{-}bundles\ on\ X \times S\}$$

Examples

$G = GL_n \leftrightarrow \mathcal{E}$ rank n vector bundle. $G = SL_n \leftrightarrow \mathcal{E}$ rank n vector bundle and $\wedge^n E \cong \mathcal{O}$ $G = PGL_n \leftrightarrow \mathcal{E}$ rank n vector bundle upto tensoring by line bundles. $G = O_n \leftrightarrow \mathcal{E}$ rank n vector bundle with a non-degenerate symmetric bi-linear form.

Example: $X = \operatorname{Spec} k$, then $\mathrm{Bun}_G(pt) = \mathbb{B}G = pt/G$.

When X is a curve, we shall discuss the geometry of Bun_G in some detail.

Fact: Bun_G is an algebraic stack locally of finite type and there exists $S \to \mathrm{Bun}_G$ smooth and surjective locally of finite type.

Theorem 12 (Weil for GL_n, folk-lore for general split G)**.** *$X_0/\mathbb{F}_q$ as before, $\mathrm{Bun}_G = \mathrm{Bun}_G(X_0)$. Then we have $G(F)\backslash G(\mathbb{A})/G(\mathscr{O}) = \mathrm{Bun}_G(\mathbb{F}_q)$ canonically where $\mathscr{O} = \prod \mathcal{O}_x$.*

Example 30. *$G = \mathbb{G}_m$, then $\mathbb{A}^\times/\mathscr{O}^\times = \bigoplus_{x\in X_0 \text{ closed.}} \mathbb{Z} = \{divisors\ on\ X_0\}$ and hence we get $F^\times\backslash\mathbb{A}^\times/\mathscr{O}^\times = \{line\ bundles\ \mathcal{L}\ on\ X_0\}$*

Given a G-bundle $\mathcal{P}$ on X_0, $\exists U \neq \phi \subset X_0$ such that $\mathcal{P}_G|_U$ is trivial.

Step 1: We claim

$$G(\mathbb{A}) = \{G\text{-bundles } \mathcal{P} \text{ on } X_0 \text{ with a trivialization on some non-empty open } U \subset X_0, \tau_\eta \text{ and a tivialization } \tau_x \text{ on } D_x = \operatorname{Spec}\mathcal{O}_x, x \in X_0 \text{ closed}\}$$

To go from LHS to RHS, cover X_0 by U and $\coprod D_x$ an *fpqc* cover and then g is the gluing data for a G-bundle trivial over U and each D_x.

Given an element of the RHS, we have $\mathcal{P}_G|_{D_x^0}$ where $D_x^0 = \operatorname{Spec} F_x$ with two trivializations τ_η and $\tau_x \iff g_x \in G(F_x)$ where $g_x\tau_x = \tau_\eta$ and note that $g_x \in G(\mathcal{O}_x)\ \forall x \in U$.

Step 2 :

From now on:

An automorphic form (everywhere un-ramified) means a function

$$\mathrm{Bun}_G)(\mathbb{F}_q) \to \mathbb{C}$$

Note that the fields are different, so not quite regular functions. So we use Grothendieck-Deligne Sheaves-Functions dictionary.

Let $Y_o/\mathbb{F}_q$ be a stack/scheme. So we have the geometric Frobenius $\Phi_Y : Y \to Y$ and $Y_0(\mathbb{F}_q) = Y^\Phi$.

Recall $\mathrm{Shv}(Y) \to \bar{\mathbb{Q}}_l$-sheaves and $\mathrm{Shv}^c(Y) \subset \mathrm{Shv}(Y)$ is the sub-category of bounded complexes with constructible cohomology $\implies$ finite dimensional fibers.

Definition 19. *A Weil sheaf on Y is a pair $(\mathcal{F} \in \mathrm{Shv}(Y), \alpha : \Phi^*\mathcal{F} \xrightarrow{\cong} \mathcal{F})$*

$(\mathcal{F}, \alpha)$ is constructible $\rightsquigarrow f_{\mathcal{F}} : Y(\mathbb{F}_q) \to \bar{\mathbb{Q}}_l$. So $y \in Y(\mathbb{F}_q) = Y^\Phi$ induces $y^*\mathcal{F} = \mathcal{F}_y \leftarrow \mathcal{F}_{\Phi(y)} = \mathcal{F}_y$ and this is a morphism in $\mathrm{Vect}^c_{\bar{\mathbb{Q}}_l}$. Taking the trace of this morphism gives $f_{\mathcal{F}}(y) \in \bar{\mathbb{Q}}_l$.

15 Sheaves-Functions Dictionary: 30/10/2025

Scribe: Youseong Lee

15.1 Sheaves–functions dictionary (cont'd)

Last time: Given a constructible Weil sheaf $(\mathcal{F}, \alpha)$ on Y, we constructed a function $f_{\mathcal{F}} : Y(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}$ via "trace of Frobenius."

Intuition with manifolds:

- $Y \to \operatorname{Spec} \mathbb{F}_q$ is a fibration from a 3-manifold M to S^1;
- the base-change $Y_{\overline{\mathbb{F}_q}} = \operatorname{Spec} \overline{\mathbb{F}_q} \times_{\operatorname{Spec} \mathbb{F}_q} Y$ can be viewed as the fiber M_0 over the basepoint $0 \in S^1$, which is a surface;
- a point $y \in Y(\mathbb{F}_q)$ is a section $S^1 \to M$.

In this 3-manifold analogue, the data of $y^*\mathcal{F}$ being a local system on S^1 is equivalent to a vector space V (over $\overline{\mathbb{Q}_\ell}$) with automorphism T, and then we can calculate the trace $\operatorname{tr}(T)$. This is what we are doing in sheaf-function construction.

Remark 31. *A Weil Lisse sheaf (σ, α) on Y is equivalent to a representation of $\mathcal{W}_Y$ (Weil group of Y). Indeed, we have the following equivalent data:*

$$\begin{aligned}
\textit{Lisse sheaf } \sigma &\Leftrightarrow \rho : \pi_1(Y_{\overline{\mathbb{F}_q}}) \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}) \\
\alpha : \Phi^*\sigma \simeq \sigma &\Leftrightarrow \textit{isomorphism } \rho \simeq \rho \circ \Phi \\
&\Leftrightarrow g \in \mathrm{GL}_n(\overline{\mathbb{Q}_\ell}) \textit{ conjugating } \sigma \textit{ and } \sigma \circ \Phi \\
&\Leftrightarrow \mathbb{Z} \ltimes \pi_1(Y_{\overline{\mathbb{F}_q}}) = \mathcal{W}_Y \to \mathrm{GL}_n(\overline{\mathbb{Q}_\ell})
\end{aligned}$$

where $\Phi : \pi_1(Y_{\overline{\mathbb{F}_q}}) \to \pi_1(Y_{\overline{\mathbb{F}_q}})$ is the geometric Frobenius.

Remark 32. *If we replace $\mathbb{Z}$ by $\hat{\mathbb{Z}} = \pi_1(\operatorname{Spec} \mathbb{F}_q)$, we get sheaves on $Y_{\mathbb{F}_q}$. Therefore, sheaves on $Y_{\mathbb{F}_q}$ induces Weil sheaves on $Y_{\overline{\mathbb{F}_q}}$.*

Example 31. *From the constant sheaf $\overline{\mathbb{Q}_{\ell,Y}}$ over Y (which is automatically a Weil sheaf,) we get the constant function 1 over $Y(\mathbb{F}_q)$.*

Example 32. *For $i : Z_0 \hookrightarrow Y_0$, the sheaf $i_*\overline{\mathbb{Q}_{\ell,Z_0}}$ gives the indicator function $\delta_{Z(\mathbb{F}_q)}$ for $Z(\mathbb{F}_q) \subseteq Y(\mathbb{F}_q)$.*

Example 33. *For $A_0/\mathbb{F}_q$ a connected algebraic group, let $\chi : A_0(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^\times$ be a character. Then we get a rank 1 local system $\mathcal{L}_\chi \in Lisse(A_0)$ via the Lang isogeny:*

$$\pi_1^{\acute{e}t}(A_0, 0) \xrightarrow{\mathcal{L}} A_0(\mathbb{F}_q) \to \overline{\mathbb{Q}_\ell}^\times.$$

Exercise 3. *Fun exercise: What is the corresponding function? χ or χ^{-1}?*

Applying the above construction:

Example 34. *For $\zeta_p \in \overline{\mathbb{Q}_\ell}^\times$ a root of unity, we get a character*

$$\chi : \mathbb{F}_q \xrightarrow{\mathrm{tr}} \mathbb{F}_p \xrightarrow{x \mapsto \zeta_p^x} \overline{\mathbb{Q}_\ell}^\times$$

and the corresponding sheaf $\mathcal{L}_\chi$ is the Artin-Schreier (AS) local system on $\mathbb{A}^1$.

Example 35. *From the character $\mathbb{F}_q^\times \simeq \mathbb{Z}_{q-1} \xrightarrow{\zeta_{q-1}} \overline{\mathbb{Q}_\ell}^\times$, we get the Kummer local system on $\mathbb{G}_m$.*

15.2 Grothendieck-Lefschetz trace formula

Big theorem:

Theorem 13 (Grothendieck-Lefschetz + ...). *Let $(\mathcal{F}, \alpha)$ be a constructible Weil sheaf. Then from α, the Frobenius acts on the (compactly supported) cohomology*

$$\mathrm{Frob} \curvearrowright C_c^*(Y, \mathcal{F}) \in \mathrm{Vect}^c_{\overline{\mathbb{Q}_\ell}}$$

where $\mathrm{Vect}^c_{\overline{\mathbb{Q}_\ell}}$ denotes the category of bounded complexes with finite dimensional cohomology (perfect complexes.) Then, we have:

$$\# := \mathrm{tr}(\mathrm{Frob}) \overset{!!!}{=} \sum_{y \in Y(\mathbb{F}_q)} f_{\mathcal{F}}(y).$$

Here we mean the supertrace for tr.

Remark 33 (Definition of compactly supported cohomology). *For open embeddings $j : U \hookrightarrow Y$, we can define $j_! : Shv(U) \to Shv(Y)$, the direct image with compact support, as the left adjoint to $j^* : Shv(Y) \to Shv(U)$. (Note: in this case, we have $j^! = j^*$.)*

Then choosing a compactification $j : Y \hookrightarrow \overline{Y}$ (which is an open embedding,) we can define

$$C_c^*(Y, \mathcal{F}) := C^*(\overline{Y}, j_! \mathcal{F})$$

where $C^*(\overline{Y}, -) = \mathrm{Hom}(\overline{\mathbb{Q}_\ell}_{,\overline{Y}}, -)$. Indeed, this is independent of the choice of the compactification.

Of course, when Y is proper, there is no distinction betweeen C^* and C_c^*.

Example 36. *Choosing $\mathcal{F} = \overline{\mathbb{Q}_{\ell}}_{,Y}$, we have $f_{\mathcal{F}} = 1$, so that*

$$\#Y(\mathbb{F}_q) = \textit{trace of } \mathrm{Frob} \curvearrowright C_c^*(Y).$$

Example for $\mathbb{P}^1$:

Remark 34 (Warm-up). *The map $f : S^1 \to S^1$ given by $z \mapsto z^n$ induces $f_* : H_1(S^1) \to H_1(S^1)$ given by $n \cdot (-) : \mathbb{Z} \to \mathbb{Z}$. Here, $H_1(S^1) \simeq \mathbb{Z}$ is given by a (noncanonical) choice of orientation on S^1. Also, f induces $f^* : H^1(S^1) \to H^1(S^1)$, which is also $n \cdot (-)$. (Surprisingly, transpose of 1×1 matrix is itself!)*

Exercise 4. *For $k = \overline{k}$, the map $\mathbb{G}_{m,k} \xrightarrow{t \mapsto t^n} \mathbb{G}_{m,k}$ induces*

$$n \cdot (-) : H_1(\mathbb{G}_m, \mathbb{Z}_\ell) \to H_1(\mathbb{G}_m, \mathbb{Z}_\ell)$$

where $H_1(\mathbb{G}_m, \mathbb{Z}_\ell) \simeq \mathbb{Z}_\ell$ noncanonically (similar to $H_1(S^1) \simeq \mathbb{Z}$.) It is same for H^1. (cf. Riemann–Hilbert formula.)

Corollary 7. *The Frobenius map $\mathbb{G}_{m,\overline{\mathbb{F}_q}} \xrightarrow{\Phi} \mathbb{G}_{m,\overline{\mathbb{F}_q}}$ induces $q \cdot (-)$ on $H^1(\mathbb{G}_m)$.*

Exercise 5. *There is canonical isomorphism $H^1(\mathbb{G}_m) \simeq H^2(\mathbb{P}^1)$ (cf. Mayer-Vietoris,) and $\Phi_{\mathbb{P}^1}$ acts as $q \cdot (-)$ on $H^2(\mathbb{P}^1)$.*

Example 37. *We have*

$$\begin{array}{rcl} C^*(\mathbb{P}^1) = & \overline{\mathbb{Q}_\ell} & \oplus \ \overline{\mathbb{Q}_\ell}[-2] \\ & \circlearrowleft & \quad \circlearrowleft \\ & \mathrm{id} & \quad\ q \end{array}$$

where the endomorphisms are the Frobenius action, so we have

$$\textit{trace} = 1 + q = \#\mathbb{P}^1(\mathbb{F}_q).$$

Example 38. *Similarly,*

$$C^*(\mathbb{P}^n) = \underset{\substack{\circlearrowleft \\ \mathrm{id}}}{\overline{\mathbb{Q}_\ell}} \oplus \ldots \oplus \underset{\substack{\circlearrowleft \\ q^n}}{\overline{\mathbb{Q}_\ell}[-2n]}$$

and we have

$$\textit{trace} = 1 + q + \ldots + q^n = \#\mathbb{P}^n(\mathbb{F}_{\mathrm{q}}).$$

Some noncompact cases:

Example 39. *We have*

$$C_c^*(\mathbb{A}^n) = \underset{\substack{\circlearrowleft \\ q^n}}{\overline{\mathbb{Q}_\ell}[-2n]}$$

via open embedding $\mathbb{A}^n \hookrightarrow \mathbb{P}^n$. Of course, trace $= q^n = \#\mathbb{A}^n(\mathbb{F}_{\mathrm{q}})$.

Remember compactly supported cohomology of $\mathbb{R}^n$? (cf. Bott-Tu, Example 1.6(c))

Example 40. *In similar ways,*

$$C_c^*(\mathbb{P}^n) = \underset{\substack{\circlearrowleft \\ \mathrm{id}}}{\overline{\mathbb{Q}_\ell}[-1]} \oplus \underset{\substack{\circlearrowleft \\ q}}{\overline{\mathbb{Q}_\ell}[-2]}$$

(For the degree 1 action, it comes from the point by excision.) Now we are taking supertrace, so we should be careful when dealing with the odd degrees:

$$\textit{trace} = -1 + q = \#\,\mathbb{G}_m(\mathbb{F}_{\mathrm{q}}).$$

Idea for proving Grothendieck-Lefschetz. For simplicity, assume that Y is proper, we have Frobenius $\Phi : Y \to Y$, and $Y(\mathbb{F}_{\mathrm{q}}) = Y^\Phi$.

Remark 35 (Lefschetz trace formula on usual cohomology)**.** *For compact (complex) manifold M, suppose $\Phi : M \to M$ has simple fixed points. That is, the diagonal $\Delta \subset M \times M$ meets the* Graph$(\Phi) \subset M \times M$ *transversally. Then the Lefschetz trace formula says that*

$$M^\Phi = \mathrm{tr}\left(\Phi \curvearrowright H^*(M)\right).$$

Also note that this formula fails for noncompact manifolds, for example, when $M = \mathbb{R}^1$ and Φ is nonzero translation.

Grothendieck found there exists some analogous trace formula. Roughly, Frob gives a contraction approaching to the fixed point.

An immediate generalization to relative version:

Remark 36. *for $\theta : Y \to Z$ over $\mathbb{F}_{\mathrm{q}}$ and a constructible Weil sheaf $(\mathcal{F}, \alpha)$ over Y, $\theta_!\mathcal{F}$ carries a Weil sheaf structure on Z by base-change. Now we have:*

$$f_{\theta_!\mathcal{F}}(z) = \sum_{\substack{y \in Y(\mathbb{F}_{\mathrm{q}}) \\ \theta(y) = z}} f_{\mathcal{F}}(y)$$

for each $z \in Z(\mathbb{F}_{\mathrm{q}})$. It is really a pushforward of function via integration along fibers. (Note that we do not worry about infinite sum because, for example, it is a sum over $\mathbb{F}_{\mathrm{q}}$-points.)

Exercise 6. *Check Grothendieck-Lefschetz (without deep theory) in the following examples:*

1. $Y = \mathbb{A}^1$, $\mathcal{F} = AS$;
2. $Y = \mathbb{G}_m$, $\mathcal{F} =$ Kummer;
3. $Y = \mathbb{G}_m$, $\mathcal{F} =$ Kummer $\otimes AS|_{\mathbb{G}_m}$. *(cf. in Deligne SGA 4.5.)*

16 Better^2 Hecke Operators: 04/11/2025

Scribe: Max Steinberg

Question from Prof. Loseu on the previous lecture: there was something called Y on which the action happened. What was Y? Answer: Y is a scheme, not necessarily smooth, but quasicompact and finite type.

16.1 Where are we?

We said that, in some sense,

$$\{\text{unramified automorphic forms for } G\} \xrightarrow{\sim} \{\text{local systems on } X\}$$

(Somehow the left-hand side is pairs (G, F) and the right-hand side is $X \supseteq \operatorname{Spec} F$). When F is $\mathbb{F}_q(X_0)$:

$$\{\overline{\mathbb{Q}}_\ell\text{-valued automorphic forms}\} = \{\mathrm{Bun}_G(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell\} \leftrightarrow \{\text{Weil local systems on } X\}$$

Then: we expect "geometric origins" for functions like $Y(\mathbb{F}_q) \to \overline{\mathbb{Q}}_\ell$. There should be some kind of "canonical Weil sheaf" on Y that has more geometric origins that gives rise to f via sheaf-functions correspondence.

Summary: $\forall X/k$ with $k = \overline{k}$, $\ell \neq \mathrm{char}(k)$, for σ a GL_n-local system on X (or $\check{G}$) we should get a ℓ-adic sheaf on Bun_G $\mathcal{F}_\sigma \in \mathsf{Shv}(\mathrm{Bun}_G)$ a canonical sheaf. It should be *so canonical* that if $X/\mathbb{F}_q$ and $\sigma \simeq \Phi^*\sigma$ a Weil structure, then we get $\mathcal{F}_\sigma \simeq \mathcal{F}_{\Phi^*\sigma}$ that comes from the pullback along the geometric Frobenius on Bun_G, i.e. a Weil structure on $\mathcal{F}_\sigma$, which then gives us an automorphic form.

16.2 Hecke Symmetries

(Version 1)
Before: for every p prime, we had a corresponding operator T_p (and T_{p^n}) acting on the space of modular forms M_k.
Now: for every point $x \in X$, we get a Hecke operator $T_x : \mathsf{Shv}(\mathrm{Bun}_{\mathrm{PGL}_2}) \to \mathsf{Shv}(\mathrm{Bun}_{\mathrm{PGL}_2})$ (and ditto for GL_2 and GL_n).

Definition 20 (standard Hecke stack)**.** *The **standard Hecke stack** at x is*

$$\begin{tikzcd} & \mathcal{H}_x \arrow[dl, "\mathcal{E}=\overleftarrow{h}"'] \arrow[dr, "\mathcal{E}'=\overrightarrow{h}"] & \\ \mathrm{Bun}_{\mathrm{GL}_n} & & \mathrm{Bun}_{\mathrm{GL}_n} \end{tikzcd}$$

Where $\mathcal{H}_x : \{\mathcal{E}, \mathcal{E}' \text{ rank } n \text{ vector bundles on } X, \mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x), \dim(\mathcal{E}'/\mathcal{E}) = 1\}$.

Remark 37. *For $\mathcal{E}$ fixed, $\mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x)$ is the same as subspaces of $\mathcal{E}(x)/\mathcal{E}$ so all possible choices are a union of the Grassmannian of $\mathcal{E}(x)/\mathcal{E}$.*

Then the Hecke stack is the same as $\mathbb{P}(\mathcal{E}(x)/\mathcal{E})$.

In terms of the geometry: the map $\mathcal{H}_x \xrightarrow{\overleftarrow{h}} \mathrm{Bun}_{\mathrm{GL}_n}$ is a $\mathbb{P}^{n-1}$-bundle.

Definition 21. $T_x : \mathsf{Shv}(\mathrm{Bun}_{\mathrm{GL}_n}) \to \mathsf{Shv}(\mathrm{Bun}_{\mathrm{GL}_n})$ *is given by* $\overrightarrow{h}_! \overleftarrow{h}^*[n-1]$ $(= \overrightarrow{h}_* \overleftarrow{h}^![-(n-1)])$*. (Other conventions exist.)*

A very very proto-version of a Hecke eigensheaf:

Definition 22. *If σ is a rank-n local system on X, we could ask for $\mathcal{F} \in \mathsf{Shv}(\mathrm{Bun}_{\mathrm{GL}_n})$ with a "canonical" isomorphism $T_x(\mathcal{F}) \to \mathcal{F} \otimes \sigma_x$, where σ_x is the fibre of σ at x.*

Can $\mathcal{F}$ be a local system? Exercise: prove $\mathcal{F} \neq 0$ cannot be a local system for $n > 1$.
Better:

Definition 23 (better standard Hecke stack[8]). *$\mathcal{H}_X$ has map $\vec{h} : \mathrm{Bun}_{\mathrm{GL}_n} \times X$ given by $(\mathcal{E}', x)$ and the fibres over X give $\mathcal{H}_x$.*

Definition 24 (better Hecke operator). *$T_X : \mathsf{Shv}(\mathrm{Bun}_{\mathrm{GL}_n}) \to \mathsf{Shv}(\mathrm{Bun}_{\mathrm{GL}_n} \times X)$ is $\vec{h}_! \overleftarrow{h}^*[n-1] = \vec{h}_* \overleftarrow{h}^![-n+?]$.*

Definition 25 (better Hecke eigensheaf). *Same but $T_X(\mathcal{F}) = \mathcal{F} \boxtimes \sigma$.*

Remark 38. *A PGL_n bundle is a rank n vector bundle up to tensoring by line bundles. The standard Hecke stack makes sense for these also.*

Definition 26 (length-i standard Hecke stack). *Instead of $\dim \mathcal{E}'/\mathcal{E} = 1$, it is i instead.*

Definition 27 (T_X^i). *Shifts are now $[i(n-i)] = \dim \mathrm{Gr}(n,i)$ and $[???]$. That is $T_X^i = \vec{h}_! \overleftarrow{h}^*[i(n-i)]$.*

Definition 28 (better[2] Hecke eigensheaf). *$T_X^i(\mathcal{F}) = \mathcal{F} \boxtimes \bigwedge^i \sigma \forall 0 < i \leq n$.*

Remark 39. *When $G = \mathrm{GL}_1$: $T_X : \mathsf{Shv}(\mathrm{Bun}_{\mathbb{G}_m}) \to \mathsf{Shv}(\mathrm{Bun}_{\mathbb{G}_m} \times X)$ which is the pullback along $X \times \mathrm{Bun}_{\mathbb{G}_m} \to \mathrm{Bun}_{\mathbb{G}_m}$ which sends $(x, \mathcal{L}) \to \mathcal{L}(-x)$.*

[8] This is the standard Hecke stack while the previous definition is the standard Hecke stack *at $x \in X$*.

17 Constant Term Functors and Whittaker Coefficients: 06/11/2025

Scribe: Minghan Sun

The goal of this and the next few lectures is to discuss "q-expansion" and Whittaker coefficients.

17.1 Constant term functions of modular forms

Recall 1. *Suppose f is a (holomorphic) modular form. Recall that f has a q-expansion*

$$f(q) = a_0 + a_1 q + a_2 q^2 + \cdots . \tag{38}$$

We have

$$a_0 = \int_{\mathbb{R}/\mathbb{Z}} f(\tau) \tag{39}$$

$$a_n = \int_{\mathbb{R}/\mathbb{Z}} f(\tau) e^{-2\pi n \tau}, \tag{40}$$

where $\mathbb{R}/\mathbb{Z}$ denotes any horizontal line segment going from the left to the right of the fundamental domain of the $SL_2(\mathbb{Z})$-action on the upper half plane.

Remark 40. *It is often advantageous to consider the constant term of the q-expansion (i.e. a_0) as different from the other coefficients a_n. For example, it is better to think of a_0 as a function out of $\mathbb{R}^{>0}$ such that*

$$a_0(y) = \int_{iy+\mathbb{R}\mathbb{Z}} f(\tau). \tag{41}$$

If f is a holomorphic modular form, then $a_0(y)$ is a constant function. However, if f is not holomorphic (e.g. if f is a Maass form), then $a_0(y)$ is not necessarily constant.

17.2 Constant term functors of automorphic forms

17.2.1 The rough idea

In the world of modular forms, the constant term function integrates over shifted copies of $\mathbb{R}/\mathbb{Z}$. We have said many times that $\mathbb{R}/\mathbb{Z}$ is analogous to $F\backslash\mathbb{A}$. As a result, it is reasonable that the constant term functor associated to an automorphic form should integrate over "shifted copies of $F\backslash\mathbb{A}$" in some precise sense which we will describe. Moreover, in the world of modular forms, the constant term functions integrates over $\mathbb{G}_a$-*shifted* copies of $\mathbb{R}/\mathbb{Z}$. So it is reasonable that the constant term functor of an automorphic form should integrate over U-shifted copies of $F\backslash\mathbb{A}$ (where U is the unipotent radical of some parabolic), since U (up to "smudging") is not that different from $\mathbb{G}_a$.

Let us illustrate the above discussion a little bit. Suppose we have (G, P, U, M) and suppose $\phi : G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \to \mathbb{C}$ is an automorphic form. We want to produce a function that gives us values of integrals of ϕ along U-shifted copies of $F\backslash\mathbb{A}$.

We have a diagram

$$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \xleftarrow{p} P(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \xrightarrow{q} M(F)U(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O}). \tag{42}$$

We can immediately define the pullback function $p^*(\phi) : P(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \to \mathbb{C}$. We can define another function $\tilde{\phi} : M(F)U(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O}) \to \mathbb{C}$ by integrating $p^*(\phi)$ along the fibers of q, i.e. for all $x \in M(F)U(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O})$, we have

$$\tilde{\phi}(x) = \int_{q^{-1}(x)} p^*(\phi). \tag{43}$$

What do the fibers of q look like? Well, each fiber is isomorphic to $P(F)\backslash M(F)U(\mathbb{A}) = U(F)\backslash U(\mathbb{A})$. So we have indeed produced a function $(\tilde{\phi})$ that gives us values of integrals of ϕ along U-shifted copies of $F\backslash\mathbb{A}$.

17.2.2 The precise definition

Now we aim to geometrize the discussion in section 17.2.1 to obtain the precise definition of the constant term functor of an automorphic form.

We have to use the following theorem:

Theorem 14 (Iwasawa Decomposition)**.** *Suppose G is a reductive algebraic group and K is a nonarchimedean local field with ring of integers $\mathcal{O}_K$. Then*

$$G(K) = G(\mathcal{O}_K) \cdot B(K). \tag{44}$$

As a result, if $F = \mathbb{F}_q(X_0)$ is a function field, then we have

$$G(\mathbb{A}) = B(\mathbb{A}) \cdot G(\mathcal{O}). \tag{45}$$

Proposition 6. *We have*

$$P(F)\backslash G(\mathbb{A})/G(\mathcal{O}) = P(F)\backslash P(\mathbb{A})/P(\mathcal{O}) \tag{46}$$

$$M(F)U(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O}) = M(F)\backslash M(\mathbb{A})/M(\mathcal{O}). \tag{47}$$

Proof. By theorem 14, we have

$$P(F)\backslash G(\mathbb{A})/G(\mathcal{O}) = P(F)\backslash P(\mathbb{A})G(\mathcal{O})/G(\mathcal{O}). \tag{48}$$

It is clear that the RHS equals $P(F)\backslash P(\mathbb{A})/P(\mathcal{O})$, as desired. The second equality in the proposition is proven similarly. □

Recall 2. *Recall that if H is a reductive algebraic group, then*

$$H(F)\backslash H(\mathbb{A})/H(\mathcal{O}) \longleftrightarrow \{\textit{points of } \mathrm{Bun}_H\}. \tag{49}$$

As a result, it makes sense to define the following.

Definition 29 (constant term functor)**.** *Suppose G is an algebraic group, P is a parabolic subgroup, and M the associated Levi subgroup. We have a diagram*

$$\mathrm{Bun}_G \xleftarrow{p} \mathrm{Bun}_P \xrightarrow{q} \mathrm{Bun}_M\,. \tag{50}$$

We define the constant term functor *of G with respect to P, denoted $(CT_P)_!$, as the functor $q_!p^*$ (note that $(CT_P)_!$ is a functor from $\mathrm{Shv}(\mathrm{Bun}_G)$ to $\mathrm{Shv}(\mathrm{Bun}_M)$).*

Definition 30 (cuspidal sheaves)**.** *Setting as in definition 29. Suppose $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$. Then we say $\mathcal{F}$ is* cuspidal *if for all parabolics $P \subseteq G$, we have $(CT_P)_!(\mathcal{F}) = 0$.*

17.3 Whittaker coefficients (function-theoretic)

17.3.1 Character for $\mathbb{A}/F$

Construction 1 (character of $\mathbb{A}/F$)**.** *We will construct a character $\psi : \mathbb{A}/F \to \mathbb{C}^\times$ which is not quite canonical.*

Choose a 1-form ω on $U \subseteq X_0$, i.e. choose a 1-form ω with no zeros or poles on U.

For all closed points $x \in X_0$, let F_x denote the Laurent series at x. We have a map $F_x \to \mathbb{C}^\times$ given by

$$f \mapsto \exp(\mathrm{tr}(\mathrm{Res}(f \cdot \omega)2\pi i)). \tag{51}$$

Collecting, we get a map $\mathbb{A} \to \mathbb{C}^\times$. This map is zero on $F \subseteq \mathbb{A}$ by the sum of residues formula.

Remark 41. *The character $\mathbb{A}/F \xrightarrow{\psi} \mathbb{C}^\times$ we just constructed is analogous to the character $\mathbb{R}/\mathbb{Z} \to \mathbb{C}^\times$ given by $\tau \mapsto e^{2\pi i \tau}$.*

17.3.2 Whittaker coefficients for GL_2

Now suppose $G = GL_2$ with $N \simeq \mathbb{G}_a$ and $f : G(F)\backslash G(\mathbb{A}) \to \mathbb{C}$ is an automorphic function. To construct the Whittaker coefficients of f, we play a similar game as section 17.2.1. Our diagram is now

$$G(F)\backslash G(\mathbb{A}) \leftarrow N(F)\backslash G(\mathbb{A}) \to (N(\mathbb{A}), \psi)\backslash G(\mathbb{A}), \tag{52}$$

where ψ is the character constructed in construction 1. Note that the rightmost space is not really defined because we don't have a notion of a quotient with a character. However, the functions on this 'space' are well-defined: they are functions on $G(\mathbb{A})$ that are ψ-eigenvalues of the $N(\mathbb{A})$ action.

To spell out the above more concretely, to calculate the Whittaker coefficients of f, we simply calculate the values of the integrals

$$\int_{N(\mathbb{A})} f(ng)\psi(n)^{-1}dn \tag{53}$$

for all $g \in G(\mathbb{A})$.

17.3.3 Whittaker coefficients for general G

For a general reductive group G, we have maps

$$N \to \prod_{\text{simple roots}} \mathbb{G}_a \xrightarrow{\text{sum}} \mathbb{G}_a. \tag{54}$$

This is called a "Whittaker" or "nondegenerate" character. We can play the same game as in the GL_2 case.

18 Geometrization of Whittaker Coefficients: 11/11/2025

Scribe: Vladyslav Zveryk

From Whittaker Coefficients to Geometrization

In this lecture, we connect the classical theory of Fourier-Whittaker coefficients of automorphic forms with its geometric analogue. Recall that last time we chose a rational 1-form ω on X which defined a map

$$\begin{tikzcd} N(\mathbb{A}) \arrow[r, "\mathrm{sum}\circ\mathrm{proj}"'] \arrow[rr, bend left, "\psi"] & \mathbb{A} \arrow[r] & \mathbb{C}^\times \end{tikzcd}$$

that led to the picture

$$G(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \leftarrow N(F)\backslash G(\mathbb{A})/G(\mathcal{O}) \rightarrow (N(\mathbb{A}),\psi)\backslash G(\mathbb{A})/G(\mathcal{O}).$$

The Whittaker coefficient of f was defined by:

$$W_f(g) = \int N(F)\backslash N(\mathbb{A}) f(ng)\psi(n)^{-1}dn, \qquad g \in N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O}).$$

1. Classical Setting (Function Fields)

Notation

- Λ is the weight lattice, $\check{\Lambda}$ is the coweight lattice.
- Λ^+ are dominant weights, $\check{\Lambda}^+$ are dominant coweights.
- Δ are roots, $\check{\Delta}$ are coroots.

We have

$$T(\mathbb{A})/T(\mathcal{O}) \cong \mathrm{Div}_{\check{\Lambda}}(X),$$

where $\mathrm{Div}_{\check{\Lambda}}(X)$ is the group of $\check{\Lambda}$-valued divisors on X, i.e., finite formal sums $D = \sum\limits_{x\in X} \check{\lambda}_x \cdot x$ with $\check{\lambda}_x \in \check{\Lambda}$.

By the Iwasawa decomposition, we have an isomorphism

$$N(\mathbb{A})\backslash G(\mathbb{A})/G(\mathcal{O}) \simeq T(\mathbb{A})/T(\mathcal{O}),$$

which leads to

Proposition 7. *Every $(N(\mathbb{A}), G(\mathcal{O}))$-double coset can be represented by a unique $\check{\Lambda}$-valued divisor.*

For a divisor $D = \sum\limits_{x\in X} \check{\lambda}_x \cdot x \in \mathrm{Div}_{\check{\Lambda}}(X)$, we denote the corresponding element in $T(\mathbb{A})$ by t^D. Concretely, it is defined as

$$t^D = (\check{\lambda}_i(t_x))_{x\in X},$$

where t_x is a chosen uniformizing parameter at $x \in X$. Clearly, this construction depends on the choice of the uniformizing parameters, but this dependence is fixed after quotiening by $T(\mathcal{O})$.

The construction says that the non-zero Whittaker coefficients are indexed by D. In fact, many of the Whittaker coefficients are zero:

Proposition 8. *Let $I(g) := \int\limits_{N(\mathbb{A})} f(n'g)\psi(n')dn'$. If there exists $n \in N(\mathbb{A})$ such that $\psi(n) \neq 1$ and $t^{-D}nt^D \in G(\mathcal{O})$, then $I(t^D) = 0$.*

Proof. Assume f is right $G(\mathcal{O})$-invariant. Let $n \in N(\mathbb{A})$ satisfy the conditions.

$$\begin{aligned}
I(t^D) &= \int\limits_{N(\mathbb{A})} f(n't^D)\psi(n')dn' \\
&= \int\limits_{N(\mathbb{A})} f(n'nt^D)\psi(n'n)dn' \quad \text{(change of variable } n' \to n'n) \\
&= \psi(n) \int\limits_{N(\mathbb{A})} f(n'(nt^D))\psi(n')dn' \\
&= \psi(n) \int\limits_{N(\mathbb{A})} f(n't^D(t^{-D}nt^D))\psi(n')dn' \\
&= \psi(n) \int\limits_{N(\mathbb{A})} f(n't^D k)\psi(n')dn' \quad \text{(where } k = t^{-D}nt^D \in G(\mathcal{O})) \\
&= \psi(n) \int\limits_{N(\mathbb{A})} f(n't^D)\psi(n')dn' \quad \text{(by } G(\mathcal{O})\text{-invariance)} \\
&= \psi(n)I(t^D)
\end{aligned}$$

Since $\psi(n) \neq 1$, we must have $I(t^D) = 0$. □

Summary: The Whittaker coefficients $W_f(t^D)$ are non-zero only when D satisfies a positivity condition. These "Whittaker cells" are indexed by $\check{\Lambda}^+$-valued divisors (dominant coweight-valued divisors).

- **Example ($G = PGL_2$):** $\check{\Lambda}^+ = \mathbb{Z}_{\geq 0}\check{\omega}_1$. The coefficients are indexed by effective divisors $D = \sum n_x \cdot x$ with $n_x \geq 0$.
- **Example ($X = \operatorname{Spec}\mathbb{Z}$):** Divisors $\sum n_p[p]$ correspond to integers $\prod p^{n_p}$. The coefficients $W_f(t^D)$ correspond to the classical Fourier coefficients $a_n(f)$ where $n \leftrightarrow D$.

2. Geometrization

We now replace classical objects with their geometric counterparts (stacks, sheaves). Let $G = PGL_2$ and Ω be the canonical bundle on X. We introduce a twist.

We have $N = \mathbb{G}_a$ and

$$N(F)\backslash N(\mathbb{A})/N(\mathcal{O}) = \mathrm{Bun}_N \simeq \{0 \to \mathcal{O}_X \to \mathcal{E} \to \mathcal{O}_X \to 0\}.$$

We consider the stack Bun_N^Ω of extensions:

$$\mathrm{Bun}_N^\Omega := \{0 \to \Omega \to \mathcal{E} \to \mathcal{O}_X \to 0\}$$

This stack is

$$\mathrm{Bun}_N^\Omega \cong \mathrm{R}\Gamma(X,\Omega)[1] \simeq H^1(X,\Omega) \times \mathbb{B}H^0(X,\Omega),$$

because its points are parametrized by $\mathrm{Ext}^1(\mathcal{O}_X,\Omega) \cong H^1(X,\Omega)$, and their automorphisms are $H^0(X,\Omega)$. By Serre duality, $H^1(X,\Omega) \cong H^0(X,\mathcal{O}_X)^* \cong \mathbb{A}^1$. This gives a map $\psi : \mathrm{Bun}_N^\Omega \to H^1(X,\Omega) \cong \mathbb{A}^1$. The picture is

$$\begin{tikzcd} N^\Omega(F)\backslash N^\Omega(\mathbb{A})/N^\Omega(\mathcal{O}) \arrow[r] \arrow[rr, bend left, "\psi"] & \mathbb{F}_q \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

Let $p_N : \mathrm{Bun}_N^\Omega \to \mathrm{Bun}_G$ be the forgetful map sending the extension $\mathcal{E}$ to the G-bundle $\mathcal{E}$. (Note that $\deg \mathcal{E} = \Omega$, so the image of this map will lie in the corresponding connected component of Bun_G).

The **basic Fourier coefficient** is the functor:

$$\begin{aligned} \mathrm{coeff}_! : \mathrm{Shv}(\mathrm{Bun}_G) &\to \mathrm{Vect} \\ \mathcal{F} &\mapsto \mathrm{R}\Gamma_c(\mathrm{Bun}_N^\Omega, p_N^! \mathcal{F} \otimes \psi^* AS). \end{aligned}$$

This is an analogue of $a_1(f)$.

Generalization (Higher Coefficients)

To geometrize the coefficients $a_n(f)$ (or $W_f(t^D)$) for an effective divisor D, we define a modified stack. Let Bun_N^D be the stack of extensions:

$$\mathrm{Bun}_N^{\Omega(-D)} := \{0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O}_X \to 0\}$$

This corresponds to $\mathrm{Ext}^1(\mathcal{O}_X, \Omega(-D)) \cong H^1(X, \Omega(-D))$.

We have a map $p_{N,D} : \mathrm{Bun}_N^D \to \mathrm{Bun}_G$ sending the extension to $\mathcal{E}$. We also have a map

$$\psi_D : \mathrm{Bun}_N^D \to H^1(X, \Omega(-D)) \to H^1(X, \Omega) \simeq \mathbb{A}^1.$$

The **generalized coefficient functor** (analogue of a_n) is:

$$\begin{aligned} \mathrm{coeff}_{D,!} : \mathrm{Shv}(\mathrm{Bun}_G) &\to \mathrm{Vect} \\ \mathcal{F} &\mapsto \mathrm{R}\Gamma_c(\mathrm{Bun}_N^D, p_{N,D}^! \mathcal{F} \otimes \psi_D^* AS). \end{aligned}$$

19 Hecke Eigenforms: 13/11/2025

Scribe: Zachary Carlini

Geometerization of Cuspidal Hecke Eigenforms

19.1 The function-theoretic story

For $f \in \mathrm{Fun}\left(\mathrm{Bun}_G(\mathbb{F}_q)\right)$ and D a divisor on X, we define:

$$C_D(f) = \int_{\mathrm{Bun}_N^{\Omega(-D)}(\mathbb{F}_q)} f(n)\psi(n)dn,$$

so the functional C_D is a decategorification of $\mathrm{coeff}_{D,!}$. This is our automorphic analog of the Fourier coefficient functional $a_n(f) = \int_{\mathbb{G}_a} f(\tau)e^{-2\pi i n\tau}d\tau$. The Langlands philosophy predicts that given a (nice) representation $\sigma : W_X \to \mathrm{SL}_2(\overline{\mathbb{Q}_l})$, there should be a corresponding automorphic form $f_\sigma \in \mathrm{Fun}\left(\mathrm{Bun}_G(\mathbb{F}_q)\right)$ such that:

- f_σ is cuspidal. This means that for every line bundle $\mathscr{L}$ on $\mathrm{Bun}_{\mathbb{G}_m}(\mathbb{F}_q)$,

$$\sum_{\{0\to\mathscr{L}\to\mathscr{E}\to\mathscr{O}_X\to 0\}/\cong} f_\sigma(\mathscr{E}) = 0.$$

- The coefficients $C_D(f)$ can be calculated as follows.
  - $C_0(f) = 1$ (f is "normalized").
  - $C_{[x]}(f) = \mathrm{tr}\left(\sigma(F_x) \curvearrowright \overline{\mathbb{Q}_l}^2\right)$, where $F_x \in W_X$ is the image of the Frobenius map under the homomorphism $W_{\{x\}} \to W_X$ induced by the inclusion $\{x\} \hookrightarrow X$.
  - For $x \in X$ and $n \geq 1$, $C_{n[x]}(f) \cdot C_{[x]}(f) = C_{(n+1)[x]}(f) + C_{(n-1)[x]}(f)$.
  - If D_1 and D_2 are disjoint divisors, then $C_{D_1+D_2}(f) = C_{D_1}(f)C_{D_2}(f)$.

  Equivalently, for a divisor $D = \sum_{x\in X} n_x[x]$, the coefficient $C_D(f)$ can be calculated as:

$$C_D(f) = \prod_{x\in X} \mathrm{tr}\left(\sigma(F_x) \curvearrowright \mathrm{Sym}^{n_x}\left(\overline{\mathbb{Q}_l}^2\right)\right).$$

Exercise 7. *Given $\sigma : W_X \to \mathrm{SL}_2\left(\overline{\mathbb{Q}_l}\right)$, there exists exactly one function $\widetilde{f_\sigma}$ on $\mathrm{Bun}_B(\mathbb{F}_q)$ (which surjects onto $\mathrm{Bun}_G(\mathbb{F}_q)$) with the correct Whittaker coefficients. Since f_σ is supposed to be defined on Bun_G, it is overdetermined.*

Next, we categorify.

19.2 The sheaf-theoretic story

Given an irreducible rank-2 local system σ on X together with a trivialization $\wedge^2\sigma \cong \underline{\overline{\mathbb{Q}_l}}$, we should be able to produce a Hecke eigensheaf $\mathscr{F}_\sigma$ on Bun_G such that:

- $\mathscr{F}_\sigma$ is cuspidal. This means that $\mathrm{CT}_!(\mathscr{F}) = 0$.
- For a divisor $D = \sum_{x\in X} n_x[x]$, we have $\mathrm{coeff}_{D,!}(\mathscr{F}_\sigma) \cong \bigotimes_{x\in X} \mathrm{Sym}^{n_x}(\sigma_x)$, at least up to a shift.

However, unlike functions, sheaves are not completely determined by their fibers at points. Therefore, we should ask for a stronger, global statement rather than just imposing conditions on $\mathrm{coeff}_{D,!}$ for each divisor individually.

Fix $d \geq 0$. The set of all divisors D of degree d has a natural geometry – it can be identified with the points of $\mathrm{Sym}^d(X)$. We can define a functor $\mathrm{coeff}_{d,!}$ such that for every point $D \in \mathrm{Sym}^d(X)$ with inclusion map $\iota_D : \{D\} \to \mathrm{Sym}^d(X)$, the following diagram commutes:

$$\begin{tikzcd} Shv(\mathrm{Bun}_G) \arrow[d, "\mathrm{coeff}_{d,!}"'] \arrow[dr, "\mathrm{coeff}_{D,!}"] & \\ Shv\left(\mathrm{Sym}^d X\right) \arrow[r, "\iota_D^*"'] & Shv\left(\{D\}\right) \end{tikzcd}$$

The natural way to do this is as follows. We have a diagram:

$$\begin{tikzcd} & \left\{D \in \mathrm{Sym}^d X, 0 \to \Omega(-D) \to \mathscr{E} \to \mathscr{O}_X \to 0\right\} \arrow[dl, "\pi_1"'] \arrow[d, "\pi_2"'] \arrow[dr, "\pi_3"] & \\ \mathrm{Bun}_G & \mathbb{A}^1 & \mathrm{Sym}^d X \end{tikzcd}$$

where for $p = (D, 0 \to \Omega(-D) \to \mathscr{E} \to \mathscr{O}_X \to 0)$, $\pi_1(p) = \mathscr{E}$, $\pi_2(p) = \psi\left(0 \to \Omega(-D) \to \mathscr{E} \to \mathscr{O}_X \in \mathrm{Bun}_N^{\Omega(-D)}\right)$, and $\pi_3(p) = D$. If $AS \in Shv(\mathbb{A}^1)$ is the Artin-Shreier sheaf, we define:

$$\mathrm{coeff}_{d,!}(\mathscr{F}) = \pi_{3!}\left(\pi_1^*(\mathscr{F}) \otimes \pi_2^*(AS)\right).$$

We can now ask for a stronger condition on $\mathscr{F}_\sigma$. Namely, for every d, we require:

$$\mathrm{coeff}_{d,!}(\mathscr{F}_\sigma) \cong \sigma^{(d)}.$$

(Recall that $\sigma^{(d)}$ is defined as $add_*(\sigma^{\boxtimes d})^{S_d}$, where $add : X^d \to \mathrm{Sym}^d X$ is the map that sends $(x_1, \ldots, x_d)$ to the divisor $[x_1] + \cdots + [x_d]$)

19.3 Other groups

So far, we have been working with $G = \mathrm{PGL}_2$, but what about other algebraic groups? We saw that in this case, the Whittaker coefficients should be indexed by $\check{\Lambda}^+$-valued divisors. Given a representation $\sigma : W_X \to \check{G}\left(\overline{\mathbb{Q}}_l\right)$, we should be able to attach a function f_σ which, for every $\check{\Lambda}^+$-valued divisor $D = \sum_{x \in X} \check{\lambda}_x[x]$, satisfies:

$$C_D(f) = \prod_{x \in X} \mathrm{tr}\left(\sigma(F_x) \curvearrowright V^{\check{\lambda}_x}\left(\overline{\mathbb{Q}}_l\right)\right).$$

Coming Up...

In the following lectures, our goal will be able to make stuff feel more concrete. We will talk about:

- The geometry of cuspidality
- Why the $\mathrm{coeff}_{d,!}(\mathscr{F})$ values determine $\mathscr{F}$ uniquely
- Hecke operators
- The geometry of Bun_G (especially when $G = \mathrm{GL}_2$, PGL_2, or maybe SL_2)

Here is a starting point. There is a surjective map $\mathrm{Bun}_{\mathrm{GL}_2} \to \mathbb{Z}$ which sends a vector bundle $\mathscr{E}$ on X to its degree. It turns out this map induces an isomorphism $\pi_0(\mathrm{Bun}_{\mathrm{GL}_2}) \xrightarrow{\sim} \mathbb{Z}$. If $x \in X$ and $\mathscr{E}$ is a vector bundle on X, the degrees of $\mathscr{E}$ and $\mathscr{E}(x)$ differ by $\mathrm{rank}\,\mathscr{E}$. In particular, the parity of a degree-2 vector bundle on X is invariant under tensoring with a line bundle. Since $\mathrm{Bun}_{\mathrm{PGL}_2} \cong \mathrm{Bun}_{\mathrm{GL}_2} / \mathrm{Bun}\,\mathbb{G}_m$, we obtain $\pi_0(\mathrm{Bun}_{\mathrm{PGL}_2}) \cong \mathbb{Z}/2\,\mathbb{Z}$.

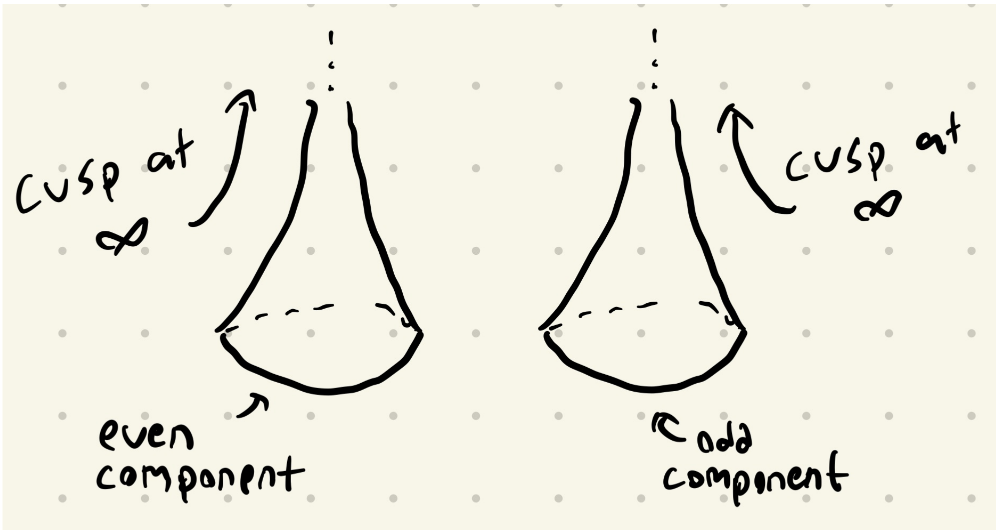


Figure 1: A cartoon of $\mathrm{Bun}_{\mathrm{PGL}_2}$

20 Semistability: 18/11/2025

Scribe: Joakim Færgeman

20.1 Definitions

For a vector bundle $\mathcal{E}$, write

$$\mu(\mathcal{E}) := \frac{\deg(\mathcal{E})}{\mathrm{rank}(\mathcal{E})}.$$

We refer to $\mu(\mathcal{E})$ as the *slope* of $\mathcal{E}$.

Definition 31. *We say $\mathcal{E}$ is* semistable *if for all subbundles $\mathcal{E}_0$, we have*

$$\mu(\mathcal{E}_0) \leq \mu(\mathcal{E}).$$

Example 41. *If $\deg(\mathcal{E}) = 0$, then $\mathcal{E}$ is semistable if and only if any subbundle $\mathcal{E}_0$ satisfies that $\deg(\mathcal{E}_0) \leq 0$.*

Example 42. *Let $\mathcal{L}$ be a line bundle. Then $\mathcal{E} := \mathcal{L} \oplus \mathcal{L}^\vee$ is semistable if and only if $\deg(\mathcal{L}) = 0$.*

Remark 42. *Observe that for a vector bundle $\mathcal{E}$ and a line bundle $\mathcal{L}$, we have*

$$\mu(\mathcal{E} \otimes \mathcal{L}) = \mu(\mathcal{E}) + \mu(\mathcal{L}).$$

It follows that if $\mathcal{E}$ is semistable, then so is $\mathcal{E} \otimes \mathcal{L}$.

Remark 43. *A vector bundle $\mathcal{E}$ on a smooth projective curve X has many subbundles in the following sense. Let $\eta_X = \mathrm{Spec}(k(X))$ be the generic point of X. Since $\mathcal{E}$ is Zariski-locally trivial, the $k(X)$-vector space*

$$V := \Gamma(\eta_X, \mathcal{E})$$

has dimension $\mathrm{rank}(\mathcal{E})$. Let $W \subset V$ be a subspace of dimension m. Consider the Grassmannian associated to $\mathcal{E}$:

$$\pi_{\mathcal{E}} : \mathrm{Gr}(\mathcal{E}) \to X$$

parametrizing subbundles of $\mathcal{E}$ of rank m. The subspace W defines a map $f : \eta_X \to \mathrm{Gr}_m(\mathcal{E})$. The map π is proper, and so by the valuative criterion for properness, the map f extends to a map

$$X \to \mathrm{Gr}_m(\mathcal{E}).$$

That is, we obtain a subbundle of $\mathcal{E}$ of rank m.

Example 43. *Suppose $\mathcal{E}$ is a rank 2 vector bundle that is not semistable. This implies that there exists a subbundle $\mathcal{L} \to \mathcal{E}$ such that $\deg(\mathcal{L}) > 0$. It is a simple exercise to see that in this case, $\mathcal{L}$ is the unique line subbundle with this property. The induced filtration*

$$0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{E}/\mathcal{L} \to 0$$

is referred to as the Harder-Narasimhan *stratification of $\mathcal{E}$.*

Let us make this last example more geometric. Let us take our structure group to be $G = \mathrm{PGL}_2$, so that the standard Borel subgroup B of G can be identified with invertible upper triangular matrices whose lower right entry is 1. For $d \in \mathbb{Z}$, consider the stack

$$\mathrm{Bun}_B^d = \{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O}_X \to 0,\ \deg(\mathcal{L}) = d\}$$

parametrizing rank 2 vector bundles $\mathcal{E}$ equipped with a filtration as above for some line subbundle $\mathcal{L}$ of degree d. Note that we have a natural map

$$\mathrm{Bun}_B^d \to \mathrm{Bun}_G$$

forgetting the filtration and remembering only the vector bundle $\mathcal{E}$. We state the following proposition whose proof we will see later in the notes:

Proposition 9.

1. *The map $\mathrm{Bun}_B^d \to \mathrm{Bun}_G$ is a locally closed embedding for $d > 0$.*
2. *$\mathrm{Bun}_B^d \to \mathrm{Bun}_G$ is smooth for $d << 0$ with the dimension of the fibers going to infinity as $d \to -\infty$.*
3. *The stack $\mathrm{Bun}_G^{\mathrm{ss}}$ parametrizing semistable G-bundles is a quasi-compact open substack of Bun_G.*

20.2 Deformation Theory

Suppose Y is a smooth variety and $y \in Y$. We have an associated tangent space $T_{Y,y}$ of dimension $\dim Y$.

20.2.1

If $\mathcal{Y}$ is a smooth stack, and $y : \mathrm{Spec}(k) \to \mathcal{Y}$, we can associate a corresponding tangent complex $T_{\mathcal{Y},y}$ which is a complex of k-vector spaces living in cohomological degrees $[-1, 0]$. Moreover, $T_{\mathcal{Y},y}$ has Euler characteristic equal to $\dim \mathcal{Y}$.

Slightly better, we may consider the quasi-coherent sheaf $T_{\mathcal{Y}}$ on $\mathcal{Y}$ whose fiber at y is $T_{\mathcal{Y},y}$.

Example 44. *Suppose H is a linear algebraic group, and let $\mathcal{Y} = BH$ be the corresponding classifying stack. Then*

$$T_{BH} = \mathfrak{h}[1] \in \mathrm{QCoh}(BH) \simeq \mathrm{Rep}(H).$$

Here, $\mathfrak{h}$ is the Lie algebra of H considered as a representation of H via the adjoint action.

Example 45. *Let X, Z be smooth stacks, and let $\mathcal{Y} = \mathrm{Maps}(X, Z)$ be the space whose T-points, for some affine test scheme T, is the groupoid $\mathrm{Maps}(X \times T, Z)$. We have an evaluation map*

$$\mathrm{ev} : \mathcal{Y} \times X \to Z$$

and a projection map $p : \mathcal{Y} \times X \to \mathcal{Y}$. Then we have:

$$T_{\mathcal{Y}} \simeq p_* \circ \mathrm{ev}^*(T_Z) \in \mathrm{QCoh}(\mathcal{Y}).$$

Here, both the pullback and the pushforward is considered in the derived *sense.*

Combining the above two examples, we obtain the following corollary for $\mathrm{Bun}_H = \mathrm{Maps}(X, BH)$:

Corollary 8. *For $\mathcal{P}_H \in \mathrm{Bun}_H$, we have:*

$$T_{\mathrm{Bun}_H, \mathcal{P}_H} \simeq R\Gamma(X, \mathfrak{h}_{\mathcal{P}_H})[1].$$

Here, $\mathfrak{h}_{\mathcal{P}_H} = \mathcal{P}_H \overset{H}{\times} \mathfrak{h}$ is the vector bundle of rank $\dim H$ obtained by twisting the bundle $\mathcal{P}_H$ by the adjoint representation.

20.2.2

In particular, if X is a smooth projective curve, we see that the tangent complex of Bun_H live in cohomological degrees $[-1, 0]$, which implies that Bun_H is smooth.

For a vector bundle $\mathcal{E}$, we write $h^i(\mathcal{E}) := \dim H^i(X, \mathcal{E})$. By smoothness of Bun_H, we have:

$$\dim_{\mathcal{P}_H} \mathrm{Bun}_H = h^1(\mathfrak{h}_{\mathcal{P}_H}) - h^0(\mathfrak{h}_{\mathcal{P}_H}) = -\chi(\mathfrak{h}_{\mathcal{P}_H}).$$

If $H = G$ is reductive, then $\mathfrak{g} \simeq \mathfrak{g}^*$ as G-representations. In particular, $\deg(\mathfrak{g}_{\mathcal{P}_G}) = 0$. By Riemann-Roch, we obtain:

$$\dim \mathrm{Bun}_G = -\chi(\mathfrak{g}_{\mathcal{P}_G}) = -\deg(\mathfrak{g}_{\mathcal{P}_G}) + \mathrm{rank}(\mathfrak{g}_{\mathcal{P}_G})(g-1) = (g-1) \cdot \dim G.$$

20.2.3

As another example, let B be the standard Borel subgroup of $G = \mathrm{PGL}_2$. For a B-bundle

$$\mathcal{P}_B = 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O}_X \to 0,$$

we have $\mathfrak{b}_{\mathcal{P}_B} = \mathcal{E}$. So:

$$\dim\ \mathrm{Bun}_B^d = -\deg(\mathfrak{b}_{\mathcal{P}_B}) + 2(g-1) = 2(g-1) - d.$$

21 Harder-Narasimhan Filtrations: 20/11/2025

Scribe: David Fang

Last time, we started looking at the HN stratification of $\mathrm{Bun}_G, G = \mathrm{PGL}_2$. In particular, we said that the map

$$\mathrm{Bun}_B^d = \{0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0 \mid \deg \mathcal{L} = d\} \xrightarrow{\mathfrak{p}_d} \mathrm{Bun}_G$$

is smooth for $d \ll 0$.

In general, if $f : \mathcal{Y} \to \mathcal{Z}$ is a map of smooth stacks, it is smooth at $y \in \mathcal{Y}$ iff on tagnent complexes the map $H^0(T_{\mathcal{Y},y}) \to H^0(T_{\mathcal{Z},f(y)})$ is surjective.

Example 46. *The map $BG \to pt$ is smooth.*

For us, we had:

$$\begin{tikzcd}
H^0(T_{\mathrm{Bun}_B,\mathcal{P}_B}) \arrow[r] \arrow[d, equal] & H^0(T_{\mathrm{Bun}_G,\mathcal{P}_G}) \arrow[d, equal] & & \\
H^1(X, \mathfrak{b}_{\mathcal{P}_B}) \arrow[r] & H^1(X, \mathfrak{g}_{\mathcal{P}_B}) \arrow[r] & H^1(X, (\mathfrak{g}/\mathfrak{b})_{\mathcal{P}_B}) \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact. But

$$\mathfrak{b} = \begin{pmatrix} * & * \\ 0 & 0 \end{pmatrix} \subseteq \begin{pmatrix} * & * \\ * & * \end{pmatrix} /\mathrm{diag} = \mathfrak{g} \implies \mathfrak{g}/\mathfrak{b} = \begin{pmatrix} 0 & 0 \\ * & 0 \end{pmatrix}$$

B acts on $\mathfrak{g}/\mathfrak{b}$ by $\mathbb{G}_m$ inverse to the standard character. Recall that $H^1(\mathcal{L}^\vee)$ is dual to $H^0(\mathcal{L} \otimes \Omega^1)$, the latter of which is 0 if $\deg \mathcal{L} \otimes \Omega^1 < 0$. In particular, we see that $\mathfrak{p}_d$ is smooth if $d < -(2g-2)$.

Remark 44. *ANy vector bundle of rank > 1 admits a line sub-bundle of degree $< -N$ for every N. Idea: take an open U such that $\mathcal{E}|_U \simeq \mathcal{O}_U^{\oplus r}$. By wha t we said last time, a line subbundle of $\mathcal{E}$ is the same as a point $\ell \in P^{r-1}(k(X))$, which is the same as a map $X \xrightarrow{f} \mathbb{P}^{r-1}$. Then we can take embeddings of arbitrarily low degree, and $\deg \mathcal{L} \approx -\deg f + const$.*

In particular we have a smooth cover

$$\coprod_{d < -(2g-2)} \mathrm{Bun}_B^d \to \mathrm{Bun}_G$$

Since Bun_B^d is an Artin stack (e.g. by checking that $\mathrm{Bun}_{\mathbb{G}_m}, \mathrm{Bun}_{\mathbb{G}_a}$ are), then Bun_G is as well. To get more information, we'll use the "Drinfeld compactification." The classical references are:

- Braverman-Gaitsgory: *Geometric Eisenstein Series...*
- Simon Schieder: *The Harder Narasimhan Stratification....*

Definition 32. *Define compactification*

$$\overline{\mathrm{Bun}}_B := \{(\mathcal{E} \text{ rank 2 v.b. }, 0 \neq \pi \in \mathrm{Hom}(\mathcal{E}, \mathcal{O}_X))\}$$

Explicitly: an S-point of $\overline{\mathrm{Bun}}_B$ is a rank 2 vector bundle $\mathcal{E}$ on $X \times S$, and a map of sheaves $\mathcal{E} \xrightarrow{pi} \mathcal{O}_{X \times S}$, such that $\forall s \in S, \pi|_{X \times \{s\}} \neq 0$.

There is a map

$$\overline{\mathrm{Bun}}_B \to \mathrm{Bun}_T = \mathrm{Bun}_{\mathbb{G}_m}, \qquad (\mathcal{E}, \pi) \mapsto \det \mathcal{E} = \bigwedge^2 \mathcal{E}.$$

We let $\mathrm{Bun}_B^d = \{(\mathcal{E}, \pi) \mid \deg \mathcal{E} = d\}$. We have maps:

$$\begin{tikzcd}
 & \mathrm{Bun}_B^d \arrow[d, hook, "\pi \text{ surj.}"', "\text{open}"] \arrow[ddl, bend right, "\mathfrak{p}_d"'] \arrow[ddr, bend left, "\mathfrak{q}_d"] & \\
 & \overline{\mathrm{Bun}}_B^d \arrow[dl, "\overline{\mathfrak{p}}_d"] \arrow[dr, "\overline{\mathfrak{q}}_d"'] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_{\mathbb{G}_m}
\end{tikzcd}$$

Proposition 10. $\overline{\mathfrak{p}}_d$ *is proper.*

We won't prove this; more or less this comes from the properness of Quot schemes. Let $\mathcal{E} \xrightarrow{\pi} \mathcal{O} \in \overline{\mathrm{Bun}}_B^d$ be a field-valued point. We can factor this as

$$\mathcal{E} \twoheadrightarrow \mathcal{O}(-D) \hookrightarrow \mathcal{O}$$

for some divisor $D \geq 0$, so we get short exact sequences

$$0 \to \mathcal{L} = \ker \pi \to \mathcal{E} \to \mathcal{O}(-D) \to 0, \qquad 0 \to \mathcal{L}(D) \to \mathcal{E}(D) \to \mathcal{O} \to 0.$$

Since we work over PGL_2, we know

$$\overline{\mathfrak{p}}_d((\mathcal{E}, \pi)) = \mathfrak{p}_{d+\deg D}(0 \to \mathcal{L}(D) \to \mathcal{E}(D) \to \mathcal{O}_X \to 0)$$

Observe the following: for $d \ll 0$, consider the maps

$$\begin{tikzcd}
\mathrm{Bun}_B^d \arrow[r, hook] \arrow[dr, "\mathfrak{p}_d"] & \overline{\mathrm{Bun}}_B^d \arrow[d, "\overline{\mathfrak{p}}_d"] \\
 & \mathrm{Bun}_G^{d \bmod 2}
\end{tikzcd}$$

Since $\mathfrak{p}_d$ is smooth, we know the image of $\overline{\mathfrak{p}_d}$ contains an open; on the other hand, $\overline{\mathfrak{p}_d}$ is proper, so $\overline{\mathfrak{p}}_d$ is surjective in this case. This also makes it easy to see that $\coprod_{d \geq -N} \mathrm{Bun}_B^d \to \mathrm{Bun}_G$ surjects for all $N \gg 0$.

Definition 33. *Define* $\mathrm{Bun}_G^{ss,even}$ *to be the complement of the images of* $\coprod_{d>0 \text{ even}} \mathrm{Bun}_B^d \to \mathrm{Bun}_G^{even}$*; this is also the complement to the image of* $\overline{\mathrm{Bun}}_B^2 \to \mathrm{Bun}_G^{even}$*, so this is open.*

The same logic as before shows that if we take U_d to be the G-bundles which are semistable or admit filtration $\{\mathcal{L} \to \mathcal{E} \to \mathcal{O}\}$ for $\deg \mathcal{L} \leq d$, then U_d is also open. We also see that if $\mathcal{P}_G \in \mathrm{Bun}_G^{ss}$ is semistable, then there is a universal constant N depending on g such that

$$\exists 0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0 \mapsto \mathcal{P}_G, \qquad -N \leq \deg \mathcal{L} \leq 0.$$

This is because there is a universal N so that $-\deg N \leq \deg \mathcal{L}$. On the other hand since $\mathcal{E}$ is semistable, $\deg \mathcal{L} \leq 0$. The same thing works for U_d, using $-N \leq \deg \mathcal{L} \leq d$. This is enough to show that U_d is quasi-compact, since it can be covered by finitely many Bun_B^d.

Remark 45. *We can show for $d > 0$ that*

$$\mathrm{Bun}_B^d \xrightarrow{\simeq} U_d \setminus U_{d-2},$$

where the latter is given the reduced substack structure. This implies that $\mathrm{Bun}_B^d \to \mathrm{Bun}_G$ *is locally closed for* $d > 0$.

General picture: Bun_G^{even} has an open quasicompact stratum of dimension $3g-3$, and then smaller strata Bun_B^d for $d > 0$ even, which has dimension $2g - 2 - d$. For $g > 0$ the above shows automatically that $\mathrm{Bun}_G^{odd,ss} \neq 0$ (since Bun_G has dimension $3g - 3$ and is nonempty).

Remark 46. *Stacky phenomenon: there are infinitely many strata, whose dimension tends to $-\infty$.*

22 Fourier Inversion: 02/12/2025

Scribe: Soumik Ghosh As before we have $G = \mathrm{PGL}_2$. For $n >> 0$, in fact for $n > 2g-2$, we have the map

$$\mathrm{Bun}_B^n \xrightarrow{\mathfrak{p}_n} \mathrm{Bun}_{\mathbb{G}_m}^n$$

If $[0 \to \mathcal{L} \to \mathcal{E} \to \mathcal{O} \to 0] \in \mathrm{Bun}_B^n$, then the short exact sequence splits non-canonically and the 'ambiguity' is given by $\Gamma(\mathcal{L})$.

We have a universal vector bundle $\mathcal{E}_n \to \mathrm{Bun}_{\mathbb{G}_m}$ with fiber $\Gamma(\mathcal{L})$ at $\mathcal{L}$. $\mathrm{Bun}_B^n = \mathbb{B}_{\mathrm{Bun}_{\mathbb{G}_m}^n} \mathcal{E}_n$.

Formally, $X \times \mathrm{Bun}_{\mathbb{G}_m}^n$ carries a universal line bundle and $\mathcal{E}_n$ is its pushforward to $\mathrm{Bun}_{\mathbb{G}_m}^n$.

Remark 47. *For a general n, the fiber is the complex $R\Gamma(\mathcal{L})[1]$.*

We have

$$\mathfrak{p}_{n!} : \mathrm{Shv}(\mathrm{Bun}_B^n) \to \mathrm{Shv}(\mathrm{Bun}_{\mathbb{G}_m}^n)$$

is an equivalence $\forall n >> 0$.

Application: $\mathcal{F} \in \mathrm{Shv}(\mathrm{Bun}_G)$ is cuspidal. ($CT_!^n(\mathcal{F}) = 0\ \forall n$). Suppose $n >> 0$. Then we have

$$CT_!^n(\mathcal{F}) = \mathfrak{q}_{n!}\mathfrak{p}_n^*\mathcal{F} = 0.$$

But $\mathfrak{q}_{n!}$ is an equivalence so $\mathfrak{p}_n^*\mathcal{F} = 0$.

So $\mathcal{F}$ is cuspidal $\implies$ $*$-restriction of $\mathcal{F}$ to Bun_B^n, a HN stratum vanishes for $n > 2g-2$.

Picture: $\mathcal{F}$ vanishes around ∞ $\iff$ $j_{n!}j_n^*\mathcal{F} = \mathcal{F}$ where $j_n : U_n \hookrightarrow \mathrm{Bun}_G$ is the inclusion of the union of strata $\leq n$.

For functions, we have $f : \mathrm{Bun}_G(\mathbb{F}_q) \to \mathbb{C}$ is cuspidal $\implies$ $\mathrm{Supp}\, f \subset U_n(\mathbb{F}_q)$ and U_n is quasi-compact $\implies$ $U_n(\mathbb{F}_q)$ is finite $\implies$ cuspidal automorphic functions are finite dimensional.

Let $CT_*^n := \mathfrak{q}_{n*}\mathfrak{p}_n^!$. Then Drinfield-Gaitsgory showed : $\forall n$ we have

$$\mathrm{inv}^* \circ CT_*^n \simeq \mathrm{CT}_!^{-n}$$

where $\mathrm{inv} : \mathrm{Bun}_{\mathbb{G}_m}^n \xrightarrow{\simeq} \mathrm{Bun}_{\mathbb{G}_m}^{-n}$ is the morphism $\mathcal{L} \to \mathcal{L}^{-1}$

Application: $\mathcal{F}$ vanishes around ∞ iff $\mathcal{F} = j_{n!}j_n^*\mathcal{F} = j_{n*}j_n^*\mathcal{F}$. (clean extension property)

22.1 Fourier Inversion

We have $\mathrm{coeff}_{d!} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Sym}^n X)$.

Goal: For $\mathcal{F}$ cuspidal, the knowledge of $\mathrm{coeff}_{d!}$ is equivalent to that of $\mathfrak{p}_{-d-c}^*(\mathcal{F})$. So if $\mathcal{F}$ is cuspidal, we can recover $\mathcal{F}|_{\mathrm{Bun}_B}^*$ from $\{\mathrm{coeff}_{d!}\}$.

We prove this using Fourier-Deligne equivalence.

Setup: V is a finite dimensional vector space over a field k. We define a functor

$$\begin{aligned} \mathrm{Shv}(V) &\to \mathrm{Shv}(V^\vee) \\ \mathcal{F} &\mapsto \mathcal{F}^\vee \end{aligned}$$

Consider

$$\begin{tikzcd} & V \times V^\vee \arrow[dl, "pr_1"'] \arrow[d, "ev"] \arrow[dr, "pr_2"] & \\ V & \mathbb{A}^1 & V^\vee \end{tikzcd}$$

Then $\mathcal{F}^\vee = pr_{2!}(pr_1^*\mathcal{F} \otimes ev^* AS)$.

Facts:

- $\mathrm{Shv}(V) \xrightarrow{\simeq} \mathrm{Shv}(V^\vee)$
- Upto shifts, we coincides with $()^!$ $()_*$ variant.

- This story works for vector bundles/schemes.

Consider

$$
\begin{tikzcd}
\left\{(\mathcal{L}, \omega \in \gamma(\mathcal{L})^\vee = \Gamma(\mathcal{L}^\vee \otimes \Omega^1)[1])\right\} = E_n^\vee \arrow[drr] & & E_n = \{(\mathcal{L}, s \in \gamma(\mathcal{L}))\} \supset E_n^o = \{(\mathcal{L}, s) : s \neq 0\} = \operatorname{Sym}^n X \arrow[d] \\
& & \operatorname{Bun}^n_{\mathbb{G}_m}
\end{tikzcd}
$$

23 More Fourier Inversion: 04/12/2025

Scribe: Youseong Lee

23.1 Fourier inversion (cont.)

Last time: Started "Fourier inversion" for $G = \mathrm{PGL}_2$.

The basic diagram where we do Fourier transform is:

$$\begin{tikzcd} \mathrm{Bun}_B^{-n+2g-2} \arrow[r, equal] \arrow[d, "\mathfrak{p}_n"] & E_n^\vee \arrow[dr] & & E_n \arrow[dl] \\ \mathrm{Bun}_G & & \mathrm{Bun}_{\mathbb{G}_m}^n & \mathrm{Sym}^n X \arrow[u, phantom, "\cup|"] \end{tikzcd}$$

where n denote the degree and

$$\begin{aligned} E_n &= \{(\mathcal{L}, s \in \Gamma(\mathcal{L}))\} \\ E_n^\vee &= \{0 \to \Omega \to \mathcal{E} \to \mathcal{L} \to 0, \text{ Line bundle } + \text{ Extension}\} \\ &= \{0 \to \Omega \otimes \mathcal{L}^{-1} \to \mathcal{E} \to \mathcal{O} \to 0\} \\ &= \mathrm{Bun}_B^{-n+2g-2} \end{aligned}$$

and $\mathrm{Sym}^n X \subseteq E_n$ is open, complement to the zero section.

Remark 48. *There is no similar complement construction to the zero section in $E_n^\vee$, because zero section is not a closed embedding in $E_n^\vee$ so that we cannot take its complement. Proof: The fiber over $\mathcal{L} \in \mathrm{Bun}_{\mathbb{G}_m}$ is*

$$R\Gamma(\mathcal{L}^\vee \otimes \Omega)[1] = \mathbb{B}H^0(\mathcal{L}^\vee \otimes \Omega) \times H^1(\mathcal{L}^\vee \otimes \Omega)$$

where $\mathrm{Spec}\, k \to \mathbb{B}G$ is not a closed embedding in general, since its fiber is G.

This diagram captures a lot of constructions before.

Claim 4. *Consider the following diagram:*

$$\begin{tikzcd} & & & \mathsf{Shv}(\mathrm{Sym}^n X) \\ \mathsf{Shv}(\mathrm{Bun}_G) \arrow[r, "\mathfrak{p}^*"] \arrow[urrr, bend left, "F_1"] \arrow[drrr, bend right, "F_2"'] & \mathsf{Shv}(\mathrm{Bun}_B) \arrow[r, "FT", "\sim"'] & & \mathsf{Shv}(\{(\mathcal{L}, s \in \Gamma(\mathcal{L}))\}) \arrow[u, "res"'] \arrow[d, "res^* \text{ to } 0"] \\ & & & \mathsf{Shv}(\mathrm{Bun}_{\mathbb{G}_m}) \end{tikzcd}$$

Then:

1. $F_1 = \mathrm{coeff}_{n,!}$
2. $F_2 = CT_!$

Proof. Claim 1: For convenience, we fix $D \in \mathrm{Sym}^n X$ and take the fiber over it, using that the $*$-fiber of $\mathrm{coeff}_{n,!}(\mathcal{F})$ under $\{D\} \hookrightarrow \mathrm{Sym}^n X$ is $\mathrm{coeff}_{D,!}(\mathcal{F})$. The image of $\{D\} \hookrightarrow \mathrm{Sym}^n X \subseteq \{(\mathcal{L}, s \in \Gamma(\mathcal{L}))\}$ is given by

$$(\mathcal{L}, s) = (\mathcal{O}(D), 1 \in \Gamma(\mathcal{O}(D)) \in E_n,$$

so the Fourier transform works along $(E_n^\vee)_{\mathcal{L}}$, the fiber over $\mathcal{L} = \mathcal{O}(D) \in \mathrm{Bun}_{\mathbb{G}_m}$, where

$$(E_n^\vee)_{\mathcal{L}} = \{0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0\} = \mathrm{Bun}_N^{\Omega(-D)}.$$

Recall the required definitions:

- $\mathrm{coeff}_{D,!} : \mathsf{Shv}(\mathrm{Bun}_G) \to \mathrm{Vect}$ maps
$$\mathcal{F} \mapsto R\Gamma_c(\mathrm{Bun}_N^{\Omega(-D)},\ p_{N,D}^!\mathcal{F} \otimes \psi_D^*(AS))$$
where
  - $\mathrm{Bun}_N^{\Omega(-D)} = \{0 \to \Omega(-D) \to \mathcal{E} \to \mathcal{O} \to 0\}$,
  - $p_{N,D} : \mathrm{Bun}_N^{\Omega(-D)} \to \mathrm{Bun}_G$ maps the extension to $\mathcal{E}$,
  - $\psi_D : \mathrm{Bun}_N^{\Omega(-D)} \to H^1(X, \Omega(-D)) \to H^1(X, \Omega) \simeq \mathbb{A}^1$
- FT maps $\mathfrak{p}_n^*\mathcal{F}$ to $pr_{2,!}(pr_1^*\mathfrak{p}_n^*\mathcal{F} \otimes ev^*AS)$, so its restriction to $\{D\}$ is
$$pr_1^*\mathfrak{p}_n^*\mathcal{F} \otimes ev^*AS \in \mathsf{Shv}(E_n^\vee \times_{\mathrm{Bun}_{\mathbb{G}_m}} E_n) \xrightarrow{res_{\mathcal{L}}^*} \mathsf{Shv}((E_n^\vee)_{\mathcal{L}}) \xrightarrow{R\Gamma_c} \mathrm{Vect}$$

Now the restriction of $\mathfrak{p}_n^*\mathcal{F}$ to $(E_n^\vee)_{\mathcal{L}}$ coincides with $p_{N,D}^!(\mathcal{F})$. Also, note that ev restricted to $(E_n)_{\mathcal{L}}^\vee$ is the same as the pullback along the Serre duality pairing
$$(E_n)_{\mathcal{L}}^\vee = \{0 \to \Omega \otimes \mathcal{L}^\vee \to \mathcal{E} \to \mathcal{O} \in 0\} = \Gamma(\mathcal{L}^\vee \otimes \Omega)[1] \simeq \Gamma(\mathcal{L})^\vee \xrightarrow{s} \mathbb{A}^1.$$
which is exactly the same with ψ_D. Therefore, we have the following commutative diagram:

$$\begin{tikzcd}
 & \mathrm{Bun}_N^{\Omega(-D)} \arrow[rr, equal] \arrow[ddl, bend right, "p_{N,D}"'] \arrow[dd, hook] & & (E_n^\vee)_{\mathcal{L}} \arrow[dl, hook', "res_{\mathcal{L}}"'] \arrow[d, "\psi_D"] \arrow[dr] & \\
 & & E_n^\vee \times_{\mathrm{Bun}_{\mathbb{G}_m}} E_n \arrow[r, "ev"'] \arrow[dl, "pr_1"] \arrow[dr, "pr_2"'] & \mathbb{A}^1 & \{D\} \arrow[dl, hook', "res_{\mathcal{L}}"'] \arrow[d, hook] \\
\mathrm{Bun}_G & E_n^\vee \arrow[l, "\mathfrak{p}_n"] \arrow[d, hook] \arrow[dr] & & E_n \arrow[dl] & \mathrm{Sym}^n X \arrow[l, hook', "res"] \\
 & \mathrm{Bun}_B \arrow[ul, "\mathfrak{p}"] & \mathrm{Bun}_{\mathbb{G}_m} & &
\end{tikzcd}$$

from which we can show that
$$\begin{aligned} p_{N,D}^!\mathcal{F} &= res_{\mathcal{L}}^* pr_1^* \mathfrak{p}_n^* \mathcal{F} \quad \forall \mathcal{F} \in \mathrm{Bun}_G \\ res_{\mathcal{L}}^* ev^* AS &= \psi_D^* AS \end{aligned}$$

and
$$\begin{aligned} res_{\mathcal{L}}^* \circ FT \circ \mathfrak{p}^*\mathcal{F} &= res_{\mathcal{L}}^* pr_{2,!}(pr_1^*\mathfrak{p}_n^*\mathcal{F} \otimes ev^*AS) \\ &= R\Gamma_c((E_n^\vee)_{\mathcal{L}},\ res_{\mathcal{L}}^* pr_1^*\mathfrak{p}_n^*\mathcal{F} \otimes res_{\mathcal{L}}^* ev^*AS) \\ &= R\Gamma_c((E_n^\vee)_{\mathcal{L}},\ p_{N,D}^!\mathcal{F} \otimes \psi_D^*AS) \\ &= \mathrm{coeff}_{D,!}(\mathcal{F}). \end{aligned}$$

So these two are the same maps.

Claim 2: The general principle is:
$$(* - \text{restriction to } 0) \circ FT = ! - \text{pushforward}.$$

In our case,
$$\begin{tikzcd}
 & \mathrm{Bun}_B \arrow[dl, "\mathfrak{p}"'] \arrow[dr, "\mathfrak{q}"] & \\
\mathrm{Bun}_G & & \mathrm{Bun}_{\mathbb{G}_m}
\end{tikzcd}$$

so that
$$CT_! = \mathfrak{q}_!\mathfrak{p}^* = res_0^* \circ FT \circ \mathfrak{p}^* = F_2.$$

$\square$

23.2 Towards Fourier theory of cuspidal sheaves

Upshot: if $\mathcal{F}$ is cuspidal, then

$$FT \circ \mathfrak{p}^* \mathcal{F} \in \mathsf{Shv}(\{(\mathcal{L}, s)\})$$

is !−extended from the complement to the zero section $= \coprod \operatorname{Sym}^n X$. This tells us that $\mathfrak{p}^*_{-n+2g-2}\mathcal{F}$ is uniquely determined by $\operatorname{coeff}_{n,!}(\mathcal{F})$.

Remark 49. *For functions, the same analysis says that cuspidal f is determined by its Whittaker coefficients.*

Recall our earlier setup:

- σ: irreducibel SL_2-local system on X
- Wanted: $\mathcal{F}_\sigma$ on Bun_G cuspidal with specified expansions.

Claim 5. *$\mathcal{F}_\sigma$ is "overdetermined," (assuming some properties.)*

Proof. For $n >> 0$, we have

$$\operatorname{Bun}_B^{-n+2g-2} \xrightarrow{\text{smooth with connected fibers}} \operatorname{Bun}_G .$$

We know that $\mathcal{F}_\sigma|_{\operatorname{Bun}_B^{-n+2g-2}}$ is FT of $\sigma^{(n)}$, so the restriction of $\mathcal{F}_\sigma$ to $\operatorname{Bun}_B^{-n+2g-2}$ is determined. On the other hand,

- One can see $\sigma^{(n)}$ is an irreducible perverse sheaf.
- So $\mathcal{F}_\sigma$ must be an irreducible perverse sheaf (here we assumed full support; otherwise pullback to Bun_B.)
- Such an irreducible perverse sheaf is determined by its restriction along a smooth map.

Therefore, restriction to $\operatorname{Bun}_B^{-n+2g-2}$ already determines the whole $\mathcal{F}_\sigma$. $\square$

Our dream statement is: Cuspidal $\mathcal{F}$ can be recovered from its Fourier expansion

$$\mathsf{Shv}(\operatorname{Bun}_G) \xrightarrow{\{\operatorname{coeff}_n\}} \prod \mathsf{Shv}(\operatorname{Sym}^n X).$$

However this is impossible because it is not fully faithful: RHS being a product kills this possibility. We need some communications between different n's.

There are various solutions to this problem. Some use Drinfeld's compactification, etc. In this class, a lazier approach will be used, with Hecke and CS. All approaches use some version of Ran's space.

23.3 Hecke Symmetries

Let G be a general reductive froup, and $\check{G}$ be its Langlands dual.

Hecke symmetry version 1.0:

$$\text{Given } V \in \operatorname{Rep}_{\check{G}},\ x \in X, \quad \text{then } T_{V,x} : \mathsf{Shv}(\operatorname{Bun}_G) \to \mathsf{Shv}(\operatorname{Bun}_G).$$

Example 47. *For $G = GL_n$ or PGL_n, their dual $\check{G} = GL_n$ or SL_n and their standard representation V, we recover our earlier construction.*

Briefly, the Hecke stack at x is

$$\mathcal{H}_x = \left\{ \left(\mathcal{P}_G, \widetilde{\mathcal{P}_G}, \tau : \mathcal{P}_G|_{X \setminus x} \simeq \widetilde{\mathcal{P}_G}|_{X \setminus x} \right) \right\}$$

where $\mathcal{P}_G, \widetilde{\mathcal{P}_G}$ are G-bundles. Hence we have

$$\begin{tikzcd} & \mathcal{H}_x \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\gamma"] \arrow[dr, "\overrightarrow{h}"] & \\ \operatorname{Bun}_G & \mathcal{H}_x^{loc} & \operatorname{Bun}_G \end{tikzcd}$$

where

$$\mathcal{H}_x^{loc} = \left\{ \begin{matrix} \mathcal{P}_G, \widetilde{\mathcal{P}_G} \text{ on } D_x, \\ \tau : \text{ iso } \mathcal{P}_G|_{\mathring{D}_x} \simeq \widetilde{\mathcal{P}_G}|_{\mathring{D}_x} \end{matrix} \right\}$$

and $D_x = \operatorname{Spec} k[[t_x]] \supseteq \mathring{D}_x = \operatorname{Spec} k((t_x))$.

Remark 50. *This is generalization of the standard Hecke stack, where we considered rank n vector bundles with $\mathcal{E} \subset \mathcal{E}' \subset \mathcal{E}(x)$.*

Remark 51 (Basic structure)**.** *Its geometric points up to isomorphism are given as $\check{\Lambda}^+$, dominant coweights for G. For example, there are locally closed substacks $\mathcal{H}_x^{loc,\check{\lambda}}$ of $\mathcal{H}_x^{loc}$ for each dominant coweight lam$\check{\text{b}}$da, and*

$$\mathcal{H}_x^{loc,\check{\mu}} \subseteq \mathcal{H}_x^{loc,\check{\lambda}} \text{ if and only if } \check{\mu} \leq \check{\lambda},$$

that is, $\check{\lambda} - \check{\mu}$ equals to some sum of simple coroots.

We have

$$\mathcal{H}_x^{loc} = L^+G \backslash Gr_{G,x}$$

where the loop groups are

$$\begin{aligned} LG &= \operatorname{Map}(\mathring{D}_x, G) \\ L^+G &= \operatorname{Map}(D_x, G) \end{aligned}$$

and $L^+G \subseteq LG$. Indeed, the definition of $Gr_{G,x}$ fixes trivialization of $\mathcal{E}' = D_x \times G$, so that there is a pullback diagram

$$\begin{tikzcd} \mathcal{H}_x^{loc} \arrow[d] & & Gr_{G,x} \arrow[ll] \arrow[d] \\ \mathbb{B}L^+G = \operatorname{Bun}_G(D_x) & & * \arrow[ll, "triv"'] \end{tikzcd}$$

which also holds for the quotient $L^+G \backslash Gr_{G,x}$ in the place of $\mathcal{H}_x^{loc}$.

Moreover, there is a natural isomorphism $Gr_G \simeq LG/L^+G$, so we may write the local Hecke stack as

$$\mathcal{H}_x^{loc} = L^+G \backslash LG / L^+G = \operatorname{Bun}(D_x) \times_{\operatorname{Bun}(\mathring{D}_x)} \operatorname{Bun}(D_x)$$

and its global version as

$$\operatorname{Bun}(X) \times_{\operatorname{Bun}(X \backslash x)} \operatorname{Bun}(X).$$

Remark 52. *There is a canonical functor*

$$\operatorname{Rep} \check{G} \to \mathsf{Shv}\, \mathcal{H}_x^{loc}, \ V^{\check{\lambda}} \mapsto IC_{\overline{\mathcal{H}}^{loc,\check{\lambda}}}$$

that maps a representation to corresponding intersection cohomology sheaf. This is called the geometric Satake functor.

24 Geometric Hecke Symmetries: 09/12/2025

Scribe: Michael Horzepa

24.1 With a fixed x

Let us recall that last time, after fixing $x \in X$, we draw the following diagram to describe the Hecke Stack at that point:

$$\begin{tikzcd} & \mathcal{H}_x \arrow[dl, "\overleftarrow{h}"'] \arrow[dr, "\overrightarrow{h}"] \arrow[d, "\gamma"] & \\ \mathrm{Bun}_G & & \mathrm{Bun}_G \\ & \mathcal{H}_x^{loc} \simeq L^+G\backslash \mathrm{Gr}_{G,x} & \end{tikzcd}$$

We recall that $L^+G\backslash \mathrm{Gr}_{G,x}$ is the points of X indexed by the dominant coweights of G. Now if we are given a highest weight representation $V^{\check{\lambda}} \in \mathrm{Rep}\,\check{G}$, we can construct the intersection cohomology sheaf $IC = IC_{\overline{\mathrm{Gr}}^{\check{\lambda}}} \in \mathrm{Shv}(L^+G\backslash \mathrm{Gr}_{G,x})$ on the local Hecke stack $\mathcal{H}_x^{loc}$. This allows us to construct the associated Hecke functor:

$$T_{V^{\check{\lambda}},x} : \mathrm{Shv}(\mathrm{Bun}_G) \to \mathrm{Shv}(\mathrm{Bun}_G)$$

$$\mathcal{F} \mapsto \overrightarrow{h}_*(\overleftarrow{h}^!(\mathcal{F}) \otimes \gamma^! IC)$$

We can summarize what's going on more explicitly in the following:

$$\begin{tikzcd} & \mathcal{H}_x \arrow[dl, "\overleftarrow{h}"] & & \\ \mathrm{Bun}_G \arrow[dr] & & \left\{ \begin{matrix} \tilde{P_G} \text{ on X + isom} \\ \tilde{P_G}|_{X\backslash x} \simeq P_G|_{X\backslash x} \end{matrix} \right\} \arrow[dl] = & \left\{ \begin{matrix} \tilde{P_G} \text{ on } D_x \text{ + isom} \\ \tilde{P_G}|_{\overset{\circ}{D}_x} \simeq P_G|_{\overset{\circ}{D}_x} \end{matrix} \right\} \\ & P_G & & \begin{matrix} \| \\ \text{Twisted version of} \\ \mathrm{Gr}_{G,x} \end{matrix} \end{tikzcd}$$

There is in turn the substack $\overline{\mathcal{H}}_x^{\check{\lambda}} \subseteq \mathcal{H}_x$ which is a twisted version of $\overline{\mathrm{Gr}}_G^{\check{\lambda}} \in \mathrm{Gr}_G$ such that

$$\gamma^! IC^{\check{\lambda}} = IC_{\overline{\mathcal{H}}_x^{\check{\lambda}}}[\text{shift}].$$

The way to think about this construction is to view the IC sheaf as a sort of measure and the pull-back-push-forward operation to be like integrating your sheaf against the measure. Viewed through this lens this is very similar to standard Hecke algebras.

We can further make sense of $T_{V,x}$ for general $V \in \mathrm{Rep}\,\check{G}$. First, we simply decompose along the irreducible representations:

$$V = \bigoplus V^{\check{\lambda}} \otimes \mathcal{M}_{\check{\lambda}}, \ \ \mathcal{M}_{\check{\lambda}} \in \mathrm{Vect}\,.$$

The $\mathcal{M}_{\check{\lambda}}$ encode the multiplicity of the representation via its dimension. The corresponding Hecke Operator is then:

$$T_{V,x} = \bigoplus T_{V^{\check{\lambda}},x} \otimes \mathcal{M}_{\check{\lambda}}.$$

A part of Geometric Satake Theory states that these operators should be compatible in the following way:

$$T_{V,x} \circ T_{W,x} \simeq T_{V \otimes W,x},$$

and even better (depending on $x \in X$):

$$\operatorname{Rep} \check{G} \curvearrowright \operatorname{Shv}(\operatorname{Bun}_G)$$

24.2 Allowing x to vary

This is nice for when $x \in X$ is fixed, but now what if we want to vary X? Then we need to upgrade our original picture for the Hecke stack:

$$\begin{tikzcd}
 & \mathcal{H}_X \arrow[dl, "\overleftarrow{h}"'] \arrow[d, "\gamma"] \arrow[dr, "\overrightarrow{h}"] \arrow[r, equal] & \left\{ \begin{array}{c} P_G, \tilde{P}_G, x \in X, \ + \text{isom} \\ P_G|_{X \setminus x} \simeq \tilde{P}_G|_{X \setminus x} \end{array} \right\} \\
\operatorname{Bun}_G & \mathcal{H}_X^{loc} = \{x \in X + \text{a pt of } L^+G \backslash \operatorname{Gr}_{G,X}\} & \operatorname{Bun}_G \times X
\end{tikzcd}$$

This is the moving points version of our earlier construction. Then once again we can use these maps to construct the Hecke functor for a representation $V \in \operatorname{Rep} \check{G}$:

$$T_V : \operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Shv}(\operatorname{Bun}_G \times X)$$

This functor has the basic property that taking the fiber at any $x \in X$ gives exactly the map $T_{V,x}$ that we constructed before. So we have just sensibly brought together all the Hecke operators for each point.

Now we can compose these new Hecke operators:

$$\begin{tikzcd}
(T_W \times Id_X)T_V : \operatorname{Shv}(\operatorname{Bun}_G) \arrow[rr] \arrow[dr, "T_V"] \arrow[d, equal] & & \operatorname{Shv}(\operatorname{Bun}_G \times X^2) \\
T_W T_V & \operatorname{Shv}(\operatorname{Bun}_G \times X) \arrow[ur, "T_W \times Id_X"] &
\end{tikzcd}$$

These operators then have the following properties:

1. $\Delta_X^! T_W T_V = T_{V \otimes W}$
2. $T_W T_V = \operatorname{swap}_{X^2} \circ T_V T_W$
3. The two properties above are compatible

In essense these enforce a commutativity constraint on our functors.

The ultimate form of what these current Hecke symmetries provide is a functor

$$\operatorname{Rep} \check{G}^I \otimes \operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Shv}(\operatorname{Bun}_G \times X^I)$$

for each $I \in \text{FinSet}$, which satisfy all the same compatibilities as above whenever you have a map of finite sets $I \to J$. But, let's say we instead only want endofunctors on the category $\operatorname{Shv}(\operatorname{Bun}_G)$. We can correct this by selecting some $\mathfrak{G} \in \operatorname{Shv}(\operatorname{Bun}_G^I)$ for $V \in \operatorname{Rep} \check{G}^I$. Then we define

$$T_{V,\mathfrak{G}} : \operatorname{Shv}(\operatorname{Bun}_G) \to \operatorname{Shv}(\operatorname{Bun}_G)$$

as the following:

$$\begin{tikzcd}
\operatorname{Shv}(\operatorname{Bun}_G) \arrow[r, "T_V"] \arrow[ddr, dashed, bend right, "T_{V,\mathfrak{G}}"] & \operatorname{Shv}(\operatorname{Bun}_G \times X^I) \arrow[d, "- \otimes^! p_2^!(\mathfrak{G})"] \\
 & \operatorname{Shv}(\operatorname{Bun}_G \times X^I) \arrow[d, "p_{1*}"] \\
 & \operatorname{Shv}(Bun_G)
\end{tikzcd}$$

Once again, the idea is that $\mathfrak{G}$ acts like a measure on X^I along which we integrate to get our final output.

Example 48. *Let $I = *$ the singleton set, and suppose we take $\mathfrak{G} = \delta_x$ the skyscraper sheaf at $x \in X$. Then*

$$T_{V,\mathfrak{G}} = T_{V,x}$$

Example 49. *Let $I = \{1, 2\}$, and suppose $V = V_1 \oplus V_2$. Then the corresponding Hecke operator should be the "integral":*

$$\int T_{V_1,x_1} T_{V_2,x_2} \mathfrak{G}_{x_1,x_2} dx_1 dx_2$$

There are various relations between these functors, and in fact the following diagram produces Hecke Functors which are canonically isomorphic:

$$\begin{tikzcd}
\operatorname{Shv}(X) \otimes \operatorname{Rep} \check{G}^2 \arrow[r, "id\otimes(\text{tensor})"] \arrow[d, "\Delta_*\otimes Id"] & \operatorname{Shv}(X) \otimes \operatorname{Rep} \check{G} \\
\operatorname{Shv}(X^2) \otimes \operatorname{Rep} \check{G}^2 &
\end{tikzcd}$$

24.3 The Category $\operatorname{Rep} \check{G}_{Ran}$ and Geometric Casselman-Shalika

So we generate a large supply of these Hecke functors, but we would like some way to organize them all. It turns out the category

$$\operatorname{Rep} \check{G}_{Ran} := \operatorname{colim}_{I \to J} \operatorname{Shv}(X^J) \otimes \operatorname{Rep} \check{G}^I,$$

is the source of them all. Let us take time to study this category itself. Firstly, the colimit is taken over the twisted arrows category over FinSet. This means for each $I \xrightarrow{f} J \in \operatorname{Tw}(\text{FinSet})$, we have an insertion map:

$$\operatorname{Shv}(X^J) \otimes \operatorname{Rep} \check{G}^I \xrightarrow{ins_f} \operatorname{Rep} \check{G}_{Ran}$$

These insertion objects are the basic objects of our category. Furthermore, for each commutative square

$$\begin{tikzcd}
I_1 \arrow[r, "f_1"] \arrow[d] & J_1 \arrow[d] \\
I_2 \arrow[r, "f_2"] & J_2
\end{tikzcd}$$

we have the following canonically commutative diagram:

$$\begin{tikzcd}
\operatorname{Shv}(X^{J_1}) \otimes \operatorname{Rep} \check{G}^{I_1} \arrow[dr, "ins_{f_1}"] \arrow[dd, "\Delta_*\otimes(\text{tensor})"'] & \\
& \operatorname{Rep} \check{G}_{Ran} \\
\operatorname{Shv}(X^{J_2}) \otimes \operatorname{Rep} \check{G}^{I_2} \arrow[ur, "ins_{f_2}"] &
\end{tikzcd}$$

There are even higher compatibilities on top of this, but we won't discuss them here.

So the new ultimate form of our Hecke operators lie in the symmetric monoidal category $\operatorname{Rep} \check{G}_{Ran}$ acting on $\operatorname{Shv}(\operatorname{Bun}_G)$ via endofunctors. The operation making the category into a monoid is the following (defined on the insertion objects, but clearly extends to the full category):

$$\operatorname{ins}_{f_1}(\mathfrak{G}_1 \otimes V_1) * \operatorname{ins}_{f_2}(\mathfrak{G}_2 \otimes V_2) = \operatorname{ins}_{f_1 \coprod f_2}((\mathfrak{G}_1 \boxtimes \mathfrak{G}_2) \otimes (V_1 \otimes V_2))$$

Here we note that $\mathfrak{G}_1 \boxtimes \mathfrak{G}_2 \in \mathrm{Shv}(X^{J_1 \coprod J_2})$ while $V_1 \otimes V_2 \in \mathrm{Rep}\, \check{G}^{I_1+I_2}$. An object in $\mathrm{Rep}\, \check{G}_{Ran}$ is a moving picture in some sense, where a single picture may consist of points of X with attached representations, but as the picture moves and points collide, the representations tensor over the target point.

Let us back up for a moment, and return to the case where $G = \mathrm{PGL}_2$. Let $D = \sum n_i x_i$ be an effective divisor on X, which gives rise to the Hecke functor T_D. Letting $V^{n_i} = \mathrm{Sym}^{n_i}(\mathrm{Std})$ (Std indicates the standard representation of PGL_2) at x_i, we obtain a correspondence

$$T_{V^{n_i},x_i} \leftrightarrow \bigoplus V^{n_i} \otimes \delta_{x_i}$$

The following theorem then allows us to allow us to use our Hecke functors to enable fourier coefficients of any degree to talk to one another:

Theorem 15. ***(Geometric Casselman-Shalika)***

$$\mathrm{coeff}_{D!}(\mathcal{F}) \simeq \mathrm{coeff}_{0!}(T_D(\mathcal{F}))$$

This generalizes to any G, where our divisor D is taken instead to be a $\check{\Lambda}^+$-valued divisor.

Remark 53. *If we take the $*$-version this also holds.*

Remark 54. *This was proven by Frenkel, Gaitgory, and Vilonen, but an alternate proof exists by Ngo.*

25 The Geometric Langlands Conjecture: 11/12/2025

Scribe: Zachary Carlini

25.1 Geometric Hecke Eigenforms

Recall from last time that we constructed an action of the monoidal category $\operatorname{Rep}\check{G}_{Ran}$ on $\operatorname{Shv}(\operatorname{Bun}_G)$. We will use this action to finally give the complete definition of a Hecke eigensheaf.

Let σ be a $\check{G}$-local system on X. Then σ is classified by a map $\pi_1(X) \to \check{G}$, so for any representation $\check{G} \curvearrowright V$, we can form the composition $\pi_1(X) \to \check{G} \to \mathrm{GL}(V)$ to obtain a map $\pi_1(X) \to \mathrm{GL}(V)$ which classifies a local system on X. Thus, we can associate to σ a (monoidal, exact) functor $\operatorname{Rep}\check{G} \to \mathsf{Shv}(X)$, $V \mapsto V_\sigma$ which sends a representation of $\check{G}$ to the resulting local system on X.

We want to construct a monoidal functor $F_\sigma : \operatorname{Rep}\check{G}_{Ran} \to \mathrm{Vect}$ which will play the role of the weight of our Hecke eigenform (recall that in the decategorified setup, an eigenvector of an algebra has a weight given by a character of that algebra, and Vect categorifies the scalars). A cartoon of F_σ is depicted in figure 2.

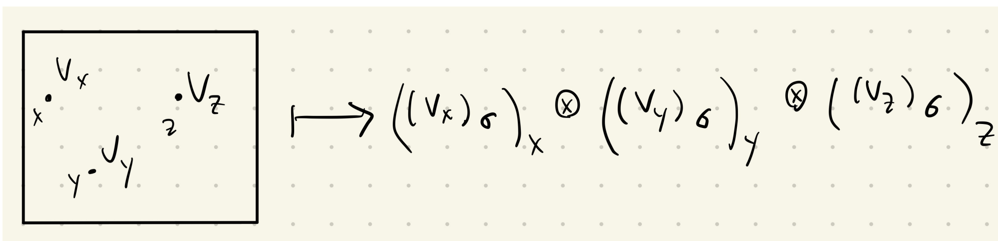


Figure 2: A cartoon of F_σ

Formally, F_σ will be the unique functor which, for each map of finite sets $f : I \to J$, makes the following diagram commute:

$$\begin{tikzcd}
\mathsf{Shv}(X^J) \otimes \operatorname{Rep}\check{G}^I \arrow[d, "\mathscr{F}\otimes V_1\otimes V_2\otimes\cdots\otimes V_k \mapsto \mathscr{F}\otimes(V_1)_\sigma\boxtimes(V_2)_\sigma\boxtimes\cdots(V_k)_\sigma"'] \arrow[dr, "ins_f"] & & \\
\mathsf{Shv}(X^J) \otimes \mathsf{Shv}(X^I) \arrow[d, "\mathscr{F}\otimes\mathscr{S}\mapsto(\Delta_f)_*(\mathscr{F})\boxtimes\mathscr{S}"'] & \operatorname{Rep}\check{G}_{Ran} \arrow[dr, "F_\sigma"] & \\
\mathsf{Shv}(X^I \times X^I) \arrow[r, "\Delta^!"'] & \mathsf{Shv}(X^I) \arrow[r, "\Gamma"'] & \mathrm{Vect},
\end{tikzcd}$$

where $\Delta_f : X^J \to X^I$ is the map which sends $(x_j)_{j\in J}$ to $(x_{f(i)})_{i\in I}$.

Example 50. *Let I be a finite set, let $f = \mathrm{id}_I$, and let $x = (x_i)_{i\in I}$ be a point in X^I. Let $(V_i)_{i\in I}$ be an I-indexed tuple of $\check{G}$-representations. Then $F_\sigma(\delta_x \otimes (V_i)_{i\in I}) \cong \bigotimes_{i\in I} x_i^!((V_i)_\sigma)$. This is what is depicted in figure 2.*

Using F_σ, we obtain an action of $\operatorname{Rep}\check{G}_{Ran}$ on Vect given by $\mathscr{F} \star V = F_\sigma(\mathscr{F}) \otimes V$. Thus, we can define a Hecke eigensheaf as follows:

Definition 34. *A Hecke eigensheaf on Bun_G with weight σ is a (exact, continuous) $\operatorname{Rep}\check{G}_{Ran}$-linear functor $\mathrm{Vect} \to \mathsf{Shv}(\operatorname{Bun}_G)$, where $\operatorname{Rep}\check{G}_{Ran}$ acts on Vect through F_σ.*

We will typically abuse notation and identify a $\operatorname{Rep}\check{G}_{Ran}$-linear functor F with the sheaf $F(\overline{\mathbb{Q}_l})$ since this determines $F(V)$ up to isomorphism for every other vector space V.

Example 51. *For $x \in X$ and V a $\check{G}$-representation, there is an element V_x in $\operatorname{Rep}\check{G}_{Ran}$ which is the image of $\partial_x \otimes V$ under the structure map $\mathsf{Shv}(X) \otimes \operatorname{Rep}\check{G} \to \operatorname{Rep}\check{G}_{Ran}$. If $\operatorname{Rep}\check{G}_{Ran}$ acts on Vect by F_σ for some local system σ, then $V_x \star \overline{\mathbb{Q}_l} \cong (V_\sigma)_x$, so for any Hecke eigensheaf $\mathscr{F}$ with weight σ, we have $V_x \star \mathscr{F} \cong (V_\sigma)_x \otimes \mathscr{F}$. But by definition, $V_x \star \mathscr{F}$ is what we were calling $T_{V,x}(\mathscr{F})$ back in section 24.1. Thus,* *definition 34 is* stronger *than the naive definition of a Hecke eigensheaf, which is just a sheaf* $\mathscr{F}$ *on* Bun_G *together with isomorphisms* $T_{V,x}(\mathscr{F}) \cong (V_\sigma)_x \otimes \mathscr{F}$ *for all* V *and* x.

Definition 35. *A* normalized *Hecke eigensheaf is a Hecke eigensheaf* $\mathscr{F}$ *together with an isomorphism* $\mathrm{coeff}_!(\mathscr{F}) \cong \overline{\mathbb{Q}_l}$.

25.2 What Is Known

25.2.1 In Characteristic 0

In characteristic 0, the following statement is known, which was called the Geometric Langlands Conjecture:

Theorem 16 (GLC I-V)**.** *For every irreducible local system* σ*, there is a unique (up to contractible choice) normalized Hecke eigensheaf* $\mathscr{F}_\sigma$ *with weight* σ*. Equivalently, if* $\mathsf{Shv}(\mathrm{Bun}_G)_\sigma$ *is the category of all Hecke eigensheaves with weight* σ*, then* $\mathrm{coeff}_! : \mathsf{Shv}(\mathrm{Bun}_G)_\sigma \to \mathrm{Vect}$ *is an equivalence.*

Who is $\mathscr{F}_\sigma$?

1. $\mathscr{F}_\sigma$ is cuspoidal.

2. There is a (useful? explicit?) formula for $\mathscr{F}_\sigma$. There is an object P_σ in $\mathrm{Rep}\,\check{G}_{Ran}$ called the Beilinson spectral projector which is the unique Hecke eigen-object of weight σ for $\mathrm{Rep}\,\check{G}_{Ran} \curvearrowright \mathrm{Rep}\,\check{G}_{Ran}$. Informally, P_σ attaches the regular representation of $\check{G}$ to every subset of X, twisted by σ. Formally, P_σ is uniquely characterized by the formula:

   $$\mathrm{Hom}(\mathbb{1}, P_\sigma \star V) \cong F_\sigma(V),$$

   where the isomorphism is natural in V. Then $\mathscr{F}_\sigma \cong P_\sigma \star \mathrm{Poinc}_!$, where $\mathrm{Poinc}_!$ is the vaccuum Poincare sheaf

   This is manifestly a Hecke eigensheaf. The non-obvious fact is that it is nonzero. The analogy to eigenvectors of algebras is the following: given a linear linear functional $\sigma : A \to k$ on an algebra A, one can define a universal eigenvector of A of weight σ to be the A module generated by a single element $\mathbb{1}$ subject to the relations $a.\mathbb{1} = \sigma(a).\mathbb{1}$ for all $a \in A$. Tensoring this module with any other A-module M gives the σ-weight space M_σ of M. One can show that σ is actually a character by producing a module M such that M_σ is nonzero. The statement that $P_\sigma \star \mathrm{Poinc}_!$ is normalized is analogous to the statement that M_σ is one-dimensional.

3. $\mathscr{F}_\sigma$ is perverse up to a normalizing shift.

4. Up to a shift, F_σ is a direct sum of simple perverse sheaves.

   Easy setup: σ is "Schurian", meaning $\mathrm{Aut}(\sigma) = Z_{\check{G}}$. In this case, the restriction of F_σ to any irreducible component of Bun_G is simple.

   In general, we have:

   $$\mathscr{F}_\sigma \cong \bigoplus_{\rho \in \mathrm{Irrep}(\mathrm{Aut}(\sigma))} \mathscr{F}_{\sigma,\rho}^{\dim \rho},$$

   where the $F_{\sigma,\rho}$ are pairwise distinct simple perverse sheaves up to a shift. There are canonical identifications:

   $$\pi_0(\mathrm{Bun}_G) \cong \pi_1(G) \cong \mathrm{Irrep}(Z_{\check{G}}),$$

   and the support of $\mathscr{F}_{\sigma,\rho}$ lies in the connected component of Bun_G corresponding to the central character of ρ.

5. The singular support of $\mathscr{F}_\sigma$, which is a subvariety of $T^*\mathrm{Bun}_G = \mathrm{Higgs}_G = \{(\rho_G, \varphi \in \mathfrak{g}_{\rho_G} \otimes \Omega_1)\}$, is contained in the subvariety $\mathrm{Nilp} = \{(\rho_G, \varphi) : \varphi \text{ is nilpotent}\}$.

$5\frac{1}{2}$. The characteristic class of $\mathscr{F}_\sigma$ is [Nilp], which allows us to compute that the generic rank of $\mathscr{F}_\sigma$ is $\prod_i d_i^{(2d_i-1)(g-1)}$, where $(d_i)_i$ are the degrees of G.

6. In the D-modules setup, $\mathscr{F}_\sigma$ can be described as the D-module freely generated by a single generator subject to some explicit relations after choosing an oper structure on σ. This result is due to [BD91].

25.2.2 In Positive Characteristic

We know that $\mathrm{coeff}_! : \mathsf{Shv}(\mathrm{Bun}_G)_\sigma \to \mathrm{Vect}$ is fully faithful, but we don't know that $\mathsf{Shv}(\mathrm{Bun}_G)_\sigma$ is nonzero. When $\mathscr{F}_\sigma$ exists, statements 1, 2, 3, 4, and 5 go through, but statement $5\frac{1}{2}$ is unknown, and statement 6 does not make sense.

When G is GL_n, PGL_n, or SL_n?, normalized Hecke eigensheaves exist for all σ.

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