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An arithmetic application of geometric Langlands

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Abstract. Vincent Lafforgue has constructed a Langlands decomposition of the space of cuspidal automorphic functions for function fields. In our joint work with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we showed that a version of the geometric Langlands conjectures yields a description of the eigenspaces of Lafforgue's decomposition in the everywhere unramified case.

In this note, we give an overview of the latter circle of ideas. We then explain how to use these methods to show that geometric Langlands implies that there are no everywhere unramified cusp forms with *trivial* Langlands parameter, addressing a question of Michael Harris.

Of some independent interest, we calculate a spectral analogue of pseudo-Eisenstein series near the trivial Langlands parameter in some explicit terms. In suitable coordinates, we find it is a product of the Weyl character formula with a zeta factor related to the curve.

1. Introduction

1.1. Background and goals.

1.1.1. In our joint works [**2**], [**3**], and [**4**] with Arinkin, Gaitsgory, Kazhdan, Rozenblyum, and Varshavsky, we formulated a conjectural spectral decomposition of unramified, compactly supported automorphic functions. This conjecture was also found by X. Zhu in [**32**].

Our spectral decomposition, inspired by V. Lafforgue's breakthroughs [**22**] and by the geometric Langlands conjecture of Beilinson-Drinfeld, is of Langlands type, but of different nature: it describes all (compactly supported) automorphic functions, not merely eigenforms, and it yields both reciprocity and functoriality statements without explicitly incorporating either into its formulation.

One major purpose of our work was to show that the spectral decomposition actually follows from an ℓ-adic version of the geometric Langlands conjecture.

1.1.2. In advertising our joint work, including in my talks at IHÉS, I have tried to argue that our conjecture yields new insights into automorphic functions that should be of interest to number theorists. Our conjecture is most manifestly satisfying around *discrete* (alias: *elliptic*) Langlands parameters (cf. Example 2.4.5.1). But I have been hard pressed to give precise, concrete consequences near other Langlands parameters.

For instance, our conjecture as is does not immediately reproduce the Arthur multiplicity formula for discrete series. Further development of the theory is needed to understand such forms.

1.1.3. With that said, the main new contribution of this note is to give a simple, concrete application of our work to automorphic functions, answering a question of Michael Harris. The assertion statement concerns the trivial Langlands parameter, which is essentially as far from discrete as possible.

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1.1.4. In addition, befitting conference proceedings, in §2 we provide some introduction to the geometric Langlands program and the circle of ideas developed in [**2**], [**3**], and [**4**]. These two parts of the paper can be read essentially independently.

The reader who is most interested in this survey material might skip ahead to §2; as that material is by its nature introductory, the emphasis of the remainder of the introduction is on the problem considered in the latter part of the paper.

1.2. Statement of the main result.

1.2.1. *Setting.* We fix $\mathbf{F}_q$ a finite field of characteristic p and let $k = \overline{\mathbf{F}}_q$ denote its algebraic closure. We let $\ell \neq p$ be a fixed prime and let $\mathbf{e}$ denote $\overline{\mathbf{Q}}_\ell$; this is the *field of coefficients* in the terminology of [**2**]. We fix $G/\mathbf{F}_q$ a split reductive group and let $\check{G}/\mathbf{e}$ denote its Langlands dual group.

Let $X_0/\mathbf{F}_q$ be a smooth, projective, and geometrically connected curve, and we let $X = X_0 \times_{\mathbf{F}_q} k$ denote its base-change to k.

We let $F = \mathbf{F}_q(X_0)$ denote the global field associated with X_0. We let $\mathbf{A}$ denote its ring of adèles and let $\mathbf{O} \subseteq \mathbf{A}$ denote the subring of integral adèles.

We let $\mathcal{A}ut_{G,c}^{\mathrm{unr}}$ denote the space of *everywhere unramified, compactly supported automorphic function* for F. By definition, this means that $\mathcal{A}ut_c^{\mathrm{unr}}$ is the vector space of functions:

$$G(F)\backslash G(\mathbf{A})/G(\mathbf{O}) \to \mathbf{e}$$

with finite support. We let $\mathcal{A}ut_{\mathrm{cusp}}^{\mathrm{unr}} \subseteq \mathcal{A}ut_c^{\mathrm{unr}}$ denote the subspace of *cuspidal* automorphic forms.[1]

1.2.2. We fix once and for all a k-point of X to use as the base-point for our fundamental groups; we omit it from the notation.[2] We let $\pi_1^{\text{ét}}(X)$ denote the étale fundamental group of X, we let $\pi_1^{\mathrm{arthm}}(X) := \pi_1^{\text{ét}}(X_0)$ denote the arithmetic fundamental group, and we let $\mathcal{W}_X := \pi_1^{\mathrm{arthm}}(X) \times_{\widehat{\mathbf{Z}}} \mathbf{Z}$ denote the Weil group of X (considered with its standard topology, so that $\pi_1^{\text{ét}}(X) \subseteq \mathcal{W}_X$ is open).

Notation 1.2.2.1. For definiteness: we always use *geometric* Frobenius conventions. So we have identified $\widehat{\mathbf{Z}} \simeq \pi_1^{\text{ét}}(\mathrm{Spec}(\mathbf{F}_q))$ with generator of $\widehat{\mathbf{Z}}$ corresponding to the *geometric Frobenius* element.

1.2.3. *Lafforgue-Langlands decomposition.* For the moment, we assume that G is semisimple to simplify the discussion. (The body of the paper works with general reductive groups.)

A *Langlands parameter* is a continuous homomorphism $\rho : \mathcal{W}_X \to \check{G}(\mathbf{e})$. A Langlands parameter is *semi-simple* if for any parabolic $\check{P} \subseteq \check{G}$ such that ρ factors through $\check{P}(\mathbf{e})$, there exists a Levi factor $\check{M} \subseteq \check{P}$ so that ρ further factors through $\check{M}(\mathbf{e})$ (see [**2**] §3.5-3.6).

We now remind that [**22**] constructed a decomposition:

$$\mathcal{A}ut_{\mathrm{cusp}}^{\mathrm{unr}} \simeq \bigoplus_{[\sigma]} \mathcal{A}ut_{\mathrm{cusp},[\sigma]}^{\mathrm{unr}} \tag{1.1}$$

where $[\sigma]$ runs over conjugacy classes of semi-simple Langlands parameters.

Remark 1.2.3.1. The above applies just as well for ramified automorphic functions. Our main results are restricted to the unramified setting, so we have chosen simply to emphasize the unramified setting throughout this text.

1.2.4. The main result in this note is the following:

Theorem A. *Let G be semi-simple (and not the trivial group). Let* $\mathrm{triv} : \mathcal{W}_X \to \check{G}(\mathbf{e})$ *denote the* trivial *Langlands parameter, i.e., the constant map with value the identity.*

Assume the restricted geometric Langlands conjecture *of* [**2**] *with its compatibility with Eisenstein series.*

Then the summand $\mathcal{A}ut_{\mathrm{cusp},[\mathrm{triv}]}^{\mathrm{unr}} \subseteq \mathcal{A}ut_{\mathrm{cusp}}^{\mathrm{unr}}$ is zero. In other words, there are no unramified cusp forms with trivial Langlands parameters.

[1] We remind the reader of the well-known fact that over function fields, cuspidal automorphic forms are *a priori* compactly supported.

[2] It is better to think in terms of the category of lisse sheaves on X, as we often do. We use $\pi_1^{\text{ét}}$ simply to make some points of our discussion more concrete.

REMARK 1.2.4.1. Although we do not emphasize this in the text, one can get by with less. Namely, according to [**2**] Corollary 14.3.5, $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ breaks up as a sum over semi-simple $\check{G}$-local systems on X. One only needs restricted geometric Langlands for the trivial local system. We expect forthcoming work to completely address this problem.

REMARK 1.2.4.2. Roughly speaking, the argument goes as follows. In §3, we discuss what Eisenstein series corresponds to on the spectral side of arithmetic Langlands. Then in §4, we provide local coordinates on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ near the trivial local system (see Theorem 4.3.3.1) and then explicitly calculate spectral Eisenstein series in these coordinates (see Theorem 4.7.2.1). From here, the result is essentially obvious (see Lemma 3.7.1.1).

REMARK 1.2.4.3. We remark that one key point of the proof of Theorem 4.7.2.1 suggests a relationship between manipulations with certain divergent series and categorical trace methods. We spell out our ideas on this subject – such as they are – in §4.6.5. This material can be read essentially independently of the rest of the paper.

1.3. Some comments.

1.3.1. *Motivation I.* The vanishing of $\mathcal{A}ut^{\mathrm{unr}}_{\mathrm{cusp},[\mathrm{triv}]}$ is an ingredient in forthcoming work of Beuzart-Plessis–Harris–Thorne studying the local Langlands correspondence for function fields via the trace formula. The above theorem leaves their results conditional on the geometric Langlands correspondence, on which a great deal of progress has been made in recent years.

1.3.2. *Motivation II.* For $G = PGL_n$, any cusp form has irreducible Langlands parameter, i.e., in this case $\mathcal{A}ut^{\mathrm{unr}}_{\mathrm{cusp},[\sigma]} = 0$ unless σ is irreducible; we refer to [**22**] Lemma 16.4 for a recent treatment (following parts of [**20**]; see the statement of [**20**] Theorem VI.9 in particular).

However, for general G the situation is more complicated: cusp forms may have reducible Langlands parameters; this is related to the failure of the Ramanujan conjecture for these cusp forms.

Still, Arthur's conjectures provide some restrictions on the σ's that may appear. First, note that there is a canonical map $\mathcal{W}_X \to \mathbf{Z} \xrightarrow{1 \mapsto q^{-1}} \mathbf{e}^\times$ that we denote $\gamma \mapsto |\gamma|$; choosing $\sqrt{q} \in \mathbf{e}^\times$, we then obtain a canonical map:

$$\begin{aligned} \mathcal{W}_X &\to \mathcal{W}_X \times SL_2(\mathbf{e}) \\ \gamma &\mapsto \left(\gamma, \begin{pmatrix} \sqrt{|\gamma|} & 0 \\ 0 & \frac{1}{\sqrt{|\gamma|}} \end{pmatrix}\right). \end{aligned} \tag{1.2}$$

Arthur's conjectures predict that $\mathcal{A}ut^{\mathrm{unr}}_{\mathrm{cusp},[\sigma]}$ will be zero except possibly when σ extends along (1.2) to an irreducible representation of $\mathcal{W}_X \times SL_2$ into $\check{G}$.[3]

1.3.3. Suppose we are given such an Arthur parameter $\sigma^\sharp : \mathcal{W}_X \times SL_2 \to \check{G}$, and suppose its restriction to $\pi_1^{\text{ét}}(X)$ is trivial, so we have a map $\sigma^\sharp : \mathbf{Z} \times SL_2 \to \check{G}$. Let $f \in \check{\mathfrak{g}}(\mathbf{e})$ denote the logarithm of $\sigma^\sharp\Big(0, \left(\begin{smallmatrix} 1 & 0 \\ 1 & 0 \end{smallmatrix}\right)\Big)$ and let $F \in \check{G}(\mathbf{e})$ denote the image of $\left(1, \begin{pmatrix} \frac{1}{\sqrt{q}} & 0 \\ 0 & \sqrt{q} \end{pmatrix}\right)$. Note that F encodes the underlying Langlands parameter of $\sigma^\sharp$, and also note that $\mathrm{Ad}_F(f) = qf$. We also note that f must be non-zero, or else $\sigma^\sharp$ will not be irreducible.

We then arrive at:

CONJECTURE 1.3.1. *Suppose $F \in \check{G}(\mathbf{e})$ is a semisimple element and let $\sigma_F : \mathcal{W}_X \to \check{G}(\mathbf{e})$ denote the corresponding Langlands parameter $\mathcal{W}_X \to \mathbf{Z} \xrightarrow{1 \mapsto F} \check{G}(\mathbf{e})$. Then $\mathcal{A}ut^{\mathrm{unr}}_{\mathrm{cusp},[F]}$ is trivial unless q is an eigenvalue of $\mathrm{Ad}_F : \check{\mathfrak{g}} \to \check{\mathfrak{g}}$.*

REMARK 1.3.3.1. Of course, q can be replaced by q^{-1} in this conjecture (apply the Cartan involution on Arthur's SL_2), which partially reflects the invariance of the conjecture under modifications of our normalizations (like geometric vs. arithmetic Frobenius).

[3] Formally, this means we have a map $SL_2 \to \check{G}$ over $\mathbf{e}$ and a continuous map $\mathcal{W}_X \to \check{G}(\mathbf{e})$ whose image commutes with the image of $SL_2(\mathbf{e}) \to \check{G}(\mathbf{e})$.

Assuming geometric Langlands, we will prove something close to this conjecture. Namely, we will show that $\mathcal{A}ut^{\mathrm{unr}}_{\mathrm{cusp},[\sigma_F]} = 0$ unless Ad_F has an eigenvalue equal to a Frobenius eigenvalue appearing in $H^1_{\text{ét}}(X) \times H^2_{\text{ét}}(X)$; as q is the Frobenius eigenvalue on $H^2_{\text{ét}}(X)$, this is somewhat stronger than the hypotheses of the conjecture. We remark that the case where F is the identity yields Theorem A.

1.4. Conventions and notation. Our hope is that this note can serve as a point of entry to the long papers [**2**], [**3**], [**4**]. We have provided some background in §2. We also refer the reader to the introduction of [**2**] for more background on the subject. We have also aimed to include precise citations whenever we use technical results from these papers with the hopes that this can help the reader navigate these works.

We generally maintain the conventions and notation of [**2**]; we refer to §0.9 in particular. We work over the *geometric* field $k = \overline{\mathbf{F}}_q$ for geometry of the curve X, its moduli stacks Bun_H of H-bundles, and so on, and we use the characteristic 0 *coefficient* field $\mathbf{e} = \overline{\mathbf{Q}}_\ell$ for geometry of local systems. Outside of §2, we have $k = \overline{\mathbf{F}}_q$. The geometry over k is classical algebraic geometry, while the algebraic geometry over $\mathbf{e}$ is derived. We use higher categorical methods. Our DG categories are assumed to be enriched over $\mathbf{e}$-vector spaces.

For an algebraic stack $\mathcal{Y}$ over k locally of finite type, we let $\mathsf{Shv}(\mathcal{Y})$ denote the DG category of $\mathbf{e}$-sheaves on $\mathcal{Y}$ (see [**15**] A.1.1 (d')). We let $\mathsf{qLisse}(\mathcal{Y}) \subseteq \mathsf{Shv}(\mathcal{Y})$ denote the subcategory of *quasi-lisse complexes* as in [**2**] Definition 1.2.6; these are objects whose (perverse, say) cohomologies are colimits of lisse sheaves (in the usual sense).

We generally refer to ∞-categories simply as *categories* to simplify the terminology.

We let $\mathsf{DGCat}_{\mathrm{cont}}$ denote the category of cocomplete (and accessible) DG categories under continuous DG functors. We consider $\mathsf{DGCat}_{\mathrm{cont}}$ as equipped with Lurie's tensor product. We let $\mathsf{Vect} \in \mathsf{DGCat}_{\mathrm{cont}}$ denote the DG category of $\mathbf{e}$-vector spaces, which is the unit for the monoidal structure.

For $\mathcal{C}$ a DG category, we let $\mathcal{C}^c$ denote its subcategory of compact objects. When $\mathcal{C}$ has a t-structure, we let $\mathcal{C}^\heartsuit$ (resp. $\mathcal{C}^{\leqslant 0}$, resp. $\mathcal{C}^{\geqslant 0}$) denote the heart of the t-structure (resp. the subcategory of connective objects, resp. the subcategory of coconnective objects).

We refer to [**15**] for background on categorical trace methods.

For H an affine algebraic group over $\mathbf{e}$, we remind that there is a moduli stack $\mathrm{LS}^{\mathrm{restr}}_H = \mathrm{LS}^{\mathrm{restr}}_H(X)$ over $\mathbf{e}$ of H-local systems (*with restricted variation*) on X. Recall that an H-local system is simply a t-exact (equivalently: right t-exact) symmetric monoidal functor $\mathsf{Rep}(H) \to \mathsf{qLisse}(X)$ (equivalently: a symmetric monoidal $\mathbf{e}$-linear functor $\mathsf{Rep}(H)^{\heartsuit,c} \to \mathsf{Lisse}(X)^\heartsuit$). Therefore, we define the stack $\mathrm{LS}^{\mathrm{restr}}_H$ to parameterize right t-exact symmetric monoidal DG functors $\mathsf{Rep}(H) \to \mathsf{qLisse}(X)$; more precisely, the $S = \mathrm{Spec}(A)$ points of $\mathrm{LS}^{\mathrm{restr}}_H$ are the groupoid of right t-exact symmetric monoidal functors $\mathsf{Rep}(H) \to A\text{-}\mathsf{mod}(\mathsf{qLisse}(X))$.

Pullback along geometric Frobenius $\mathrm{Frob}_X : X \to X$ defines a map $\mathrm{LS}^{\mathrm{restr}}_H \to \mathrm{LS}^{\mathrm{restr}}_H$ that we also call *Frobenius*.[4] Its Frobenius fixed points are by definition the stack $\mathrm{LS}^{\mathrm{arthm}}_H$, which (tautologically) parameterizes right t-exact symmetric monoidal functors from $\mathsf{Rep}(H)$ to quasi-lisse *Weil* sheaves on X.

Finally, we always assume $p = \mathrm{char}(k)$ satisfies the (mild) assumptions from [**2**] §14.4.1.

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2. AGKRRV theory

We begin with a general overview of the works [**2**] and [**4**] and some of the background material.

These works are admittedly technical. We do not intend here to provide an overview of each bit of the technical background needed for those works. However, we have tried at least to explain why certain technical issues arise (e.g., the need for derived algebraic geometry). But in this vein, we freely appeal to

[4]Note that this map is a map of stacks over $\mathbf{e}$. The Frobenius for $\mathrm{LS}^{\mathrm{restr}}_H$ can be thought of as a non-abelian/non-linear version of Frobenius acting on the ($\mathbf{e}$-vector space of) étale cohomology of X.

foundational ideas in the subject that may not be familiar to all readers: stacks, DG categories, D-modules, ℓ-adic sheaves, and IndCoh stand out. Although these subjects are technical and not always widely known, these days there are many references (and generous experts), and we think the interested reader should readily find resources to pursue their interest in the background material that comes up in the discussion.

This section is structured as follows. First, in §2.1, we explain a bit how someone interested in automorphic functions should regard about the de Rham (or D-module) geometric Langlands conjecture, and we highlight some nice pleasant features of the latter subject in comparison with the former. In §2.3, we explain the restricted geometric Langlands correspondence; one side involves moduli theory for ℓ-adic local sheaves, which we explain in §2.2. In §2.4, we explain how the story develops working over finite fields, when Frobenius is considered. Finally, in §2.5, we describe how our main arithmetic result (from [**4**]) is proved, emphasizing the key role played by Xue's work on sheaves of shtuka cohomologies.

2.1. Arithmetic and geometric Langlands.

2.1.1. *Arithmetic Langlands.* Conventional arithmetic Langlands concerns *automorphic representations*, which by definition are certain *irreducible* representations appearing in a suitable space of automorphic functions.

A crude (and perhaps vulgar) form of the Langlands philosophy predicts that automorphic representations for G correspond to Langlands parameters for $\check{G}$. There are corrections that are not quite our emphasis here: Arthur parameters should be used, L-packets appear, for number fields there is not a suitable definition of Langlands parameter (or Langlands group), and so on. What *is* our emphasis is the atomic nature of the conjecture: the basic objects are irreducible subquotients of a space of functions, not the function space itself.

2.1.2. *Geometric Langlands.* By contrast, the conventional form of the *geometric* Langlands conjecture predicts that:

$$\mathsf{D\text{-}mod}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}}) \approx \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}}). \tag{2.1}$$

In the above formula, $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}$ is a suitable enlargement of QCoh defined by Arinkin-Gaitsgory and discussed a little more in §2.1.11.

Since the conference primarily concerns *arithmetic* aspects of the Langlands program, we digress for some time to explain some starting features of the *geometric* setting, including the notation used above and some ways of thinking about the main objects that appear there.

This form of the geometric Langlands conjecture is due to Beilinson-Drinfeld and Arinkin-Gaitsgory, see [**1**] and [**13**] for an introduction to this circle of ideas. We will refer to it as the *de Rham* geometric Langlands conjecture because the theory of D-modules remembers de Rham's cohomology groups.

2.1.3. The input for geometric Langlands conjecture is a smooth projective curve X/k for a fixed field k. We assume k is algebraically closed to simplify certain points, although this is not fundamentally essential in the de Rham setting.

Then Bun_G is the space of G-bundles on X. More specifically, $\mathrm{Bun}_G = \mathrm{Bun}_G(X)$ is a *stack* whose functor of points is given by:

$$\mathrm{Bun}_G(S) := \mathrm{Hom}(X \times S, \mathbf{B}G) = \{G\text{–bundles on } X \times S\}$$

where S is an affine scheme and $\mathbf{B}G$ is the classifying stack of G. It is standard that Bun_G is a smooth algebraic stack locally of finite type, although it is not quasi-compact.

A lovely formula due to Weil[5] says that:

$$\mathrm{Bun}_G(k) = G(F)\backslash G(\mathbf{A})/G(\mathbf{O}) \tag{2.2}$$

[5] This formula is essentially obvious once one knows that G-bundles are Zariski (not merely étale) locally trivial on smooth projective curves. For GL_n, this follows from descent. For other groups, this is Steinberg's theorem.

We remark here that the theorem also holds over finite fields, as is often implicitly taken for granted in the subject. For simply-connected G, this is a theorem of Harder [**19**]. In general, one takes a surjection $G' \to G$ with G' having simply-connected derived group and $\mathrm{Ker}(G' \to G)$ being a (connected) torus; then G-bundles on X lift to G' (by class field theoretic Brauer group considerations), so we are reduced to Harder's theorem.

with $\mathbf{O} = \prod_{x \in X(k)} O_x$ the ring of *integral adèles* (for O_x the ring of Taylor series based at $x \in X(k)$), $\mathbf{A} = \operatorname{colim}_{S \subseteq X(k) \text{ finite}} \left(\prod_{x \in S} \operatorname{Frac}(O_x) \times \prod_{x \notin S} O_x \right)$ the similarly defined ring of adèles, and $F = k(X)$ the field of rational functions on X. Therefore, we can think of Bun_G as a geometric avatar of the double quotient space where unramified automorphic functions would live (if we replaced k by a finite field).

2.1.4. A foundational analogy in geometric representation theory says that when k is of characteristic zero, the category of D-modules on a stack $\mathcal{Y}$ behaves like the space of functions on the set $\mathcal{Y}(\mathbf{F}_q)$ of $\mathbf{F}_q$-points of $\mathcal{Y}$, if such a thing makes sense.[6]

There are several justifications of this idea. First, for $k = \mathbf{C}$, some D-modules are related to constructible sheaves by the Riemann-Hilbert correspondence, which are in turn related to étale sheaves by the Riemann existence theorem, which for $k = \mathbf{F}_q$ are in turn related to functions by the Grothendieck-Deligne sheaves-functions correspondence.[7]

Alternatively, one can imagine that D-modules encode linear systems of differential equations whose solutions define functions (or distributions) on $\mathcal{Y}(\mathbf{C})$, which are analogous to functions on $\mathcal{Y}(\mathbf{F}_q)$ for different reasons.

In practice, it is important in this analogy to work with *all* D-modules on $\mathcal{Y}$. For example, the *Mellin transform* in this setting is an equivalence $\mathsf{D\text{-}mod}(\mathbf{G}_m) \simeq \mathsf{QCoh}(\mathbf{A}^1/\mathbf{Z})$; it can be thought of as a simplified toy model geometric Langlands-style equivalences. Under the Mellin transform, neither holonomic nor regular holonomic objects on the left hand side have reasonable descriptions on the right hand side. One takes this as a sign that one should work with the category of "all" D-modules in geometric representation theory rather than a constructible sort of subcategory.

Moreover, by [**24**], for $G = \mathbf{G}_m$, the equivalence (2.1) does not come from an equivalence of abelian categories; that is, it is necessary to work with *derived* categories in this analogy. Per the modern understanding, we use *DG* categories in the homotopical formalism of ∞-categories; we generally abide by the convention that our DG categories should have all direct sums and functors between them should be linear, exact, and preserve direct sums. The advantage of the homotopical formalism is that it *eases* the foundational burdens of the subject by introducing algebraic tools – we speak can fluently of monoidal categories, module categories, tensor products, and so on most readily in this language.

Example 2.1.4.1. Per the previous discussion, one considers $\mathsf{D\text{-}mod}(\mathrm{Bun}_G)$ as analogous to the space $\mathcal{A}ut_c^{\mathrm{unr}}$ of unramified automorphic functions.

2.1.5. Let us pause a bit further to discuss the analogy between categories and vector spaces further.

The origin can be thought of as follows: for $\mathcal{Y}/\overline{\mathbf{F}}_q$ defined over $\mathbf{F}_q$, a constructible Weil étale sheaf $\mathcal{F}$ on $\mathcal{Y}$ gives rise to a function on $\mathcal{Y}(\mathbf{F}_q)$ by taking the trace of Frobenius on the fibers at rational points, giving a fairly general procedure for producing functions from sheaves. This is the usual source of the analogy between sheaves and functions.

One can say that functions on a space form a vector space, while sheaves on a space form a category, so vector spaces (of functions) categorify to categories (of sheaves).

2.1.6. One can make the previous discussion more precise.

Fundamentally, the source of functions in the previous discussion was that if we have a (finite-dimensional) vector space V with a linear transformation $T : V \to V$, we can form $\mathrm{tr}_V(T)$ to obtain a number.

Similarly, for a (dualizable DG) category $\mathcal{C}$ with endofunctor $T : \mathcal{C} \to \mathcal{C}$, there is a trace $\mathrm{tr}_{\mathcal{C}}(T) \in \mathsf{Vect}$ associated to this datum; we refer to [**15**] for a detailed discussion of this construction. We wish to highlight that – besides (maybe serious) psychological barriers around categories – the general construction is quite formal and mirrors the usual theory of traces.

2.1.7. We now turn to more closely interpreting the geometric Langlands equivalence.

[6]For example, the reader can imagine $\mathcal{Y}$ is defined over $\mathbf{Z}[1/N]$ for N prime to q. But I would encourage the reader not to be so literal-minded on this point.

[7]We refer to the first sections of [**28**] "Applications de la formule des traces aux sommes trigonométriques" for background on this notion.

The space $\mathrm{LS}^{\mathrm{dR}}_{\check{G}} = \mathrm{LS}^{\mathrm{dR}}_{\check{G}}(X)$ is the moduli stack of *de Rham* $\check{G}$-local systems on X. In the field, the stack $\mathrm{LS}^{\mathrm{dR}}_{H}$ (for H/k an affine algebraic group) is conventionally defined as having S-points:

$$\mathrm{LS}^{\mathrm{dR}}_{H}(S) := \mathrm{Hom}(X_{\mathrm{dR}} \times S, \mathbf{B}\check{G})$$

where X_{dR} is the de Rham space of X. It would be too digressive here to discuss the de Rham space in detail, but its key point is that $\mathsf{QCoh}(X_{\mathrm{dR}}) = \mathsf{D}\text{-}\mathsf{mod}(X)$.

Less conventionally, one can proceed as follows. First, at the level of k-points: what is an H-local system supposed to be? We could take $\mathsf{qLisse}^{\mathrm{dR}}(X) \subseteq \mathsf{D}\text{-}\mathsf{mod}(X)$ to be the subcategory of objects each of whose cohomologies is a colimit of local systems, i.e., vector bundles with connections; this is a suitable derived category of *lisse* D-modules, but we call them *quasi-lisse* to adhere to conventions from [**2**].

Then a de Rham H-local system is essentially a symmetric monoidal functor $\mathsf{Rep}(H) \to \mathsf{qLisse}^{\mathrm{dR}}(X)$; this is not quite right since for $H = \mathbf{G}_m$, such a datum is a tensor-invertible object σ of $\mathsf{qLisse}^{\mathrm{dR}}(X)$, i.e., a *cohomologically shifted* line bundle with connection; to remove that ambiguity, we refine our definition by asking that our functor be right t-exact after a cohomological shift [1] (to account for the fact that the functor sends the trivial representation to the constant sheaf $\mathbf{e}_X$, which is in degree $\dim(X) = 1$). .[8,9]

We note that this definition then behaves essentially as expected: a de Rham GL_n-local system is a rank n vector bundle on X with connection; a de Rham SO_n-local system is a rank n vector bundle $\mathcal{E}$ with connection ∇ and non-degenerate symmetric pairing $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{E} \to \mathcal{O}_X$ preserving the connections; a de Rham Sp_{2n}-local system is similar, but the non-degenerate pairing is anti-symmetric; a de Rham G_2-local system is an octonion bundle with connection; and so on.

This Tannakian definition of local systems – which is perhaps the simplest way to define local systems for general algebraic groups – adapts to give S-points for $\mathrm{LS}^{\mathrm{dR}}_{H}$: its S-points are symmetric monoidal functors:

$$\mathsf{Rep}(H) \to \mathsf{D}\text{-}\mathsf{mod}(X) \otimes \mathsf{QCoh}(S) \tag{2.3}$$

that are right t-exact up to shift by $\dim(X) = 1$.

We suggest the reader turn refer to [**2**] §4.1 for further related discussion. We briefly note that any symmetric monoidal functor $\mathsf{Rep}(H) \to \mathsf{D}\text{-}\mathsf{mod}(X)$ lands in $\mathsf{qLisse}^{\mathrm{dR}}(X)$, although (2.3) will not generally map into $\mathsf{qLisse}^{\mathrm{dR}}(X) \otimes \mathsf{QCoh}(S)$.

2.1.8. Needless to say: for a number theorist, $\mathrm{LS}^{\mathrm{dR}}_{\check{G}}$ is thought of as a moduli stack of Galois parameters.

Indeed, by (a very easy form of) the Riemann-Hilbert correspondence, for $k = \mathbf{C}$, there is an analytic identification of $\check{G}$-local systems with homomorphisms $\rho : \pi_1(X(\mathbf{C})) \to \check{G}(\mathbf{C})$ up to conjugation (although this does not work naively in S-families).

2.1.9. The next key piece of structure in the geometric Langlands conjecture is the *spectral action*.

This is the action of the monoidal category $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{dR}}_{\check{G}})$ on $\mathsf{D}\text{-}\mathsf{mod}(\mathrm{Bun}_G)$ constructed in [**?**] by Drinfeld-Gaitsgory. According to *loc. cit.*, this action is uniquely characterized by its compatibility with (a suitably strong version of) the Hecke action on $\mathsf{D}\text{-}\mathsf{mod}(\mathrm{Bun}_G)$.

Here we refer to [**?**] §1.1 for a discussion of the uniqueness and [**?**] §1.5 for the precise connection to Hecke functors.

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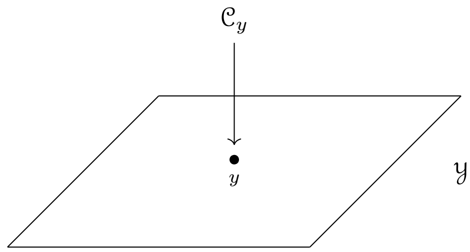
[8] Of course, this issue arises only because of our insistence to work with derived categories, which the reader may take issue with. In §2.1.12, we explain that it is necessary to use derived algebraic geometry in the story we are telling, so our affine test schemes S should also be derived; the derived category $\mathsf{QCoh}(S)$ is sensitive to derived geometry but not the abelian category $\mathsf{QCoh}(S)^{\heartsuit}$ is not.

In anticipation of these issues, we have made a pedagogical choice to stick with derived categories and right t-exact (up to shift) functors.

[9] The reader might ask: right t-exact functors $\mathsf{Rep}(H) \to \mathsf{qLisse}^{\mathrm{dR}}(X)$ are also t-exact (up to shift in both cases), so why write "right" at all? The reason is that we will soon generalize this setting in (2.3). Say H is the trivial group there, so we are talking about *the* symmetric monoidal functor $\mathsf{Vect} \to \mathsf{QCoh}(S)$, which sends k to $\mathbf{e}_X \boxtimes \mathcal{O}_S$ for $\mathbf{e}_X$ the constant sheaf. After shifting by $\dim(X)$, the latter is always connective (i.e., in cohomological degrees $\leqslant 0$), but it is only in the heart of the t-structure if S is a *classical* scheme, i.e., an object of "usual" algebraic geometry and not derived algebraic geometry.

2.1.10. The spectral action can be visualized as follows.

Let $\mathcal{Y}$ be any stack and suppose $\mathcal{C}$ is a module category for $\mathsf{QCoh}(\mathcal{Y})$. We draw this as a category fibered over $\mathcal{Y}$:



Here the fiber $\mathcal{C}_y$ at y is defined as:

$$\mathcal{C}_y := \mathcal{C} \underset{\mathsf{QCoh}(\mathcal{Y})}{\otimes} \mathsf{Vect}$$

if y is a k-point; if it is an A-point, replace Vect with A-mod. Heuristically, we might write $\mathcal{C} = \int_{y \in \mathcal{Y}} \mathcal{C}_y \, dy$. This formalism was studied in great detail in [**14**].

So informally, the Drinfeld-Gaitsgory spectral action says that the category $\mathsf{D\text{-}mod}(\mathrm{Bun}_G)$ *fibers* over $\mathrm{LS}_{\check{G}}^{\mathrm{dR}}$, and that this structure is canonically defined by Hecke functors. Therefore, the *existence* of the spectral action can be interpreted as a (categorical) *reciprocity law* for the category of automorphic sheaves (a phrase that means $\mathsf{D\text{-}mod}(\mathrm{Bun}_G)$, at least in this de Rham context).

By definition, the fiber $\mathsf{D\text{-}mod}(\mathrm{Bun}_G)_\sigma$ of $\mathsf{D\text{-}mod}(\mathrm{Bun}_G)$ at a point $\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{dR}}$ is the category of *Hecke eigensheaves* with eigenvalue σ.

2.1.11. In the heuristic formula $\mathcal{C} = \int_{y \in \mathcal{Y}} \mathcal{C}_y \, dy$ above, we imagine that we have a category-valued measure $\mathcal{C}_y \, dy$ on $\mathcal{Y}$. In cases of interest, we may wish to calculate it.

This is the job of the full geometric Langlands conjecture. The $\mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}})$-module category $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}})$ encodes an analogue of Plancherel measure under this metaphor.

Here the category $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}})$ of *ind-coherent sheaves with nilpotent singular support* was defined in [**1**] and has been the subject of wide study in the field since then. It is a modification of QCoh of geometric nature that reflects something about the singularities of $\mathrm{LS}_{\check{G}}^{\mathrm{dR}}$. We refer to [**1**] for an introduction to this subject. We use $\mathcal{N}ilp^{\mathrm{spec}} \subseteq T^*[-1]\,\mathrm{LS}_{\check{G}}^{\mathrm{dR}}$ to denote the *spectral* nilpotent cone, remarking that it is often denoted simply as $\mathcal{N}ilp$ in many other references.

Because irreducible $\check{G}$-local systems do not support non-zero nilpotent horizontal sections of their adjoint bundles, we have:

$$\mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{dR,irred}}) \underset{\mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}})}{\otimes} \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{dR}}) = \mathsf{QCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{dR,irred}})$$

(see [**1**] Proposition 13.3.3 for more details).

Under our analogy, this means that Plancherel measure is constant on $\mathrm{LS}_{\check{G}}^{\mathrm{dR,irred}}$ with value Vect. Near reducible local systems, there is a correction relating to nilpotent horizontal sections of the adjoint bundle, which are avatars here of Arthur's SL_2.

2.1.12. We now give a quick example illustrating some basic technical points.

Suppose $X = \mathbf{P}^1$ and $G = \mathbf{G}_m$. Then $\mathrm{Bun}_{\mathbf{G}_m}$ parameterizes line bundles on $\mathbf{P}^1$, so is isomorphic to $\mathbf{Z} \times \mathbf{BG}_m$: the $\mathbf{Z}$-factor parametrizes degrees of line bundles while the $\mathbf{BG}_m$-factor encodes the fact that every line bundle on $\mathbf{P}^1$ has automorphism group $\mathbf{G}_m$ (suitably understood in S-families).

Therefore, $\mathsf{D\text{-}mod}(\mathrm{Bun}_{\mathbf{G}_m}(\mathbf{P}^1)) = \prod_{n \in \mathbf{Z}} \mathsf{D\text{-}mod}(\mathbf{BG}_m)$.

The category $\mathsf{D\text{-}mod}(\mathbf{BG}_m)$ can be calculated quite explicitly. Let $\pi : \mathrm{Spec}(k) \to \mathbf{BG}_m$ be the structure map, which we remind is a smooth covering. The functor $\pi^! : \mathsf{D\text{-}mod}(\mathbf{BG}_m) \to \mathsf{D\text{-}mod}(\mathrm{Spec}(k)) = \mathsf{Vect}$ is evidently conservative and admits a left adjoint $\pi_!$. By base-change, the endofunctor $\pi^!\pi_!$ of Vect is given by tensoring with $\mathrm{C}_{\mathrm{dR}}(\mathbf{G}_m)$, the de Rham homology of $\mathbf{G}_m$. Moreover, by a simple form of Barr-Beck, this endofunctor $\pi^!\pi_!$ has a natural monad structure corresponding to the algebra structure on $\mathrm{C}_{\mathrm{dR}}(\mathbf{G}_m)$

coming from the group structure on $\mathbf{G}_m$; moreover, the induced functor $\mathsf{D}\text{-}\mathsf{mod}(\mathbf{BG}_m) \to \mathrm{C}_{\mathrm{dR}}(\mathbf{G}_m)$ is an equivalence.

Finally, of course, $\mathrm{C}_{\mathrm{dR}}(\mathbf{G}_m)$ is a *DG algebra* which is a square-zero extension of k by a single generator η in cohomological degree -1 (aliases: a symmetric algebra on a generator in degree -1; a homologically graded exterior algebra on one generator) — this is just reflecting the elementary fact that homology of the circle is 1-dimensional in degrees 0 and 1.

So we have:

$$\mathsf{D}\text{-}\mathsf{mod}(\mathrm{Bun}_{\mathbf{G}_m}(\mathbf{P}^1)) = \prod_{n\in\mathbf{Z}} (k \times k\eta)\text{-}\mathsf{mod}. \tag{2.4}$$

Naively, $\mathbf{P}^1$ is a simply-connected, so has no non-trivial local systems, so one might expect $\mathrm{LS}^{\mathrm{dR}}_{\check{G}}(\mathbf{P}^1)$ to equal $\mathbf{B}\check{G}$ (reflecting the non-trivial automorphism group of a trivial local system). For $\check{G} = \mathbf{G}_m$, we would have $\mathsf{QCoh}(\mathbf{BG}_m) = \mathsf{Rep}(\mathbf{G}_m) = \prod_{n\in\mathbf{Z}} \mathsf{Vect}$, which is close to (2.4), but missing the generator in degree -1.

In fact, this is because we were too naive. The *derived* stack $\mathrm{LS}^{\mathrm{dR}}_{\check{G}}(\mathbf{P}^1)$ equals[10] $\mathbf{B}\check{G} \times_{\check{\mathfrak{g}}/\check{G}} \mathbf{B}\check{G}$, which for $\check{G} = \mathbf{G}_m$ is just $\mathbf{BG}_m \times (0 \times_{\mathbf{A}^1} 0)$. Here it is important the fiber products be taken *in the sense of derived algebraic geometry.* Then we find $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{dR}}_{\mathbf{G}_m}(\mathbf{P}^1)) = \prod_{n\in\mathbf{Z}} \mathsf{QCoh}(0 \times_{\mathbf{A}^1} 0)$. Finally, we note that $0 \times_{\mathbf{A}^1} 0$ is Spec of $k \otimes_{k[t]} k$ (the tensor product being derived, i.e., including the information of the groups $\mathrm{Tor}_i^{k[t]}(k,k)$), which is the same square-zero extension $\mathrm{C}_{\mathrm{dR}}(\mathbf{G}_m)$ we saw before.

We remark that the underlying classical stack recovers our naive conception of LS from before.

Alternatively, one can see the utility of derived algebraic geometry as follows. For general X and $\check{G}$, standard arguments say that the tangent space of $\mathrm{LS}^{\mathrm{dR}}_{\check{G}}$ at a $\check{G}$-local system σ is $H^1_{\mathrm{dR}}(X, \check{\mathfrak{g}}_\sigma)$, the first de Rham cohomology with coefficients in the adjoint local system of σ. More generally, we should expect the tangent *complex* to be $\mathrm{C}_{\mathrm{dR}}(X, \check{\mathfrak{g}}_\sigma)[1]$. As the above example illustrates, this formula is only possible in general when $\mathrm{LS}^{\mathrm{dR}}_{\check{G}}$ is interpreted as a *derived* stack.

In summary: we use derived algebraic geometry in the spectral side of geometric Langlands *because it produces right answers* (unlike classical algebraic geometry) and because *it yields more manageable infinitesimal geometry* of moduli spaces.

2.1.13. *Conclusion.* Above, we briefly discussed arithmetic Langlands and gave a lengthier introduction to (de Rham) geometric Langlands.

There is a key difference, which §2.1.1 already hints at: in arithmetic Langlands, we study *atomic* objects (irreducible representations), whereas in *geometric Langlands* we study molecular objects (an analogue of the space of automorphic functions). One may compare the situation with the Fourier theory on the circle S^1: the *atomic* theory says (necessarily unitary) characters of S^1 are in bijection with $\mathbf{Z}$, but the actual Fourier theory says $L^2(S^1)$ is a direct integral over $\mathbf{Z}$ of 1-dimensional Hilbert spaces (i.e., $L^2(\mathbf{Z})$). In the automorphic theory, an analogue of the latter would be desirable, but the former is all we can access.

One starting point for [**2**] is an attempt to resolve this discrepancy, at least for function fields, at least in the everywhere unramified case. In the end, we end up with an arithmetic perspective closer to the geometric Langlands conjecture.

I wish to emphasize: our work is not the only one working on bridging this gap; [**32**] and [**11**] are closely related efforts, and we all were inspired by V. Lafforgue's breakthroughs [**22**].

2.2. Local systems with restricted variation: an introduction. There is an old desire to have some kind of geometric Langlands for ℓ-adic sheaves instead of D-modules. One side is easier to imagine: we should consider (certain) ℓ-adic sheaves on Bun_G instead of D-modules on Bun_G. The spectral side (i.e., the LS-side) has been less clear, but the relevant geometry was developed in [**2**]. We now summarize the story.

2.2.1. *What is the problem?* Suppose now that k is an algebraically closed field and X is a smooth projective curve over k. We let $\mathbf{e} := \overline{\mathbf{Q}}_\ell$.

We wish to imitate the general geometric Langlands story, but understanding local systems as lisse étale $\mathbf{e}$-sheaves rather than vector bundles with connection. What goes wrong?

[10] This formula comes e.g. from thinking of a local system on $\mathbf{P}^1$ as a pair of local systems on the two standard open $\mathbf{A}^1$'s in $\mathbf{P}^1$ with an isomorphism on their intersection $\mathbf{A}^1 \backslash 0$. We note that $\mathbf{A}^1$ is *contractible*, not just simply-connected.

First, let us be maximally optimistic: we wish to have a stack $\mathrm{LS}_{\check{G}}^{\text{ét}} = \mathrm{LS}_{\check{G}}^{\text{ét}}(X)$ that behaves like our earlier stack $\mathrm{LS}_{\check{G}}^{\mathrm{dR}}$ from before. Suitably understood, its points should be $\check{G}$-local systems, i.e., right t-exact symmetric monoidal functors $\mathsf{Rep}(\check{G}) \to \mathsf{qLisse}(X)$ – here $\mathsf{qLisse}(X) \subseteq \mathsf{Shv}(X) = \mathsf{Shv}^{\text{ét}}(X)$ is understood in the étale sense, as in §1.4.

As a preliminary step, note that the automorphism group of the trivial $\check{G}$-local system on X is $\check{G}(\mathbf{e})$. This suggests that $\mathrm{LS}_{\check{G}}^{\text{ét}}$ should be defined over the field $\mathbf{e}$ and contain a copy of $\mathbf{B}\check{G}$ corresponding to the trivial local system.

Remark 2.2.1.1. Here we see a basic bifurcation in the algebraic geometry of the geometric Langlands; some objects, like X, G, Bun_G, etc. live over the *ground field* k, but spectral objects, like $\check{G}$, $\mathrm{LS}_{\check{G}}^{(?)}$, etc. live over the *coefficient field* $\mathbf{e}$. For the de Rham theory, the coefficient field is the ground field and this distinction can be ignored.

2.2.2. Now let us suppose k has characteristic 0 and X has genus $g > 0$.

Ignoring technical issues (stackyness, derived structures), we might first guess that $\mathrm{LS}_{\mathbf{G}_m}^{\text{ét}}$ would be something like $\mathbf{G}_m^{2g}$ over the field $\mathbf{e}$. After all, the étale fundamental group of X has abelianization $\widehat{\mathbf{Z}}^{2g}$. Moreover, one can see that (neglecting the same technical issues), the stack $\mathrm{LS}_{\mathbf{G}_m}^{\mathrm{dR}}$ over $\mathbf{C}$ is *complex analytically* isomorphic to $\mathbf{G}_m^{2g}(\times \mathbf{B}G_m \times 0 \times_{\mathbf{A}^1} 0)$.

However, the difference between $\mathbf{Z}$ and $\widehat{\mathbf{Z}}$ is key here. In point of fact, continuous homomorphisms $\widehat{\mathbf{Z}}^{2g} \to \mathbf{e}^\times$ are indexed by points in $(O_{\mathbf{e}}^\times)^{2g}$ where $O_{\mathbf{e}} \subseteq \mathbf{e}$ is its usual valuation subring of integral elements. In other words, our hope $\mathrm{LS}_{\mathbf{G}_m}^{\text{ét}} = \mathbf{G}_m^{2g}(\times \mathbf{B}G_m \times 0 \times_{\mathbf{A}^1} 0) \approx \mathbf{G}_m^{2g}$ was too naive: the right hand side has too many points over $\mathbf{e}$!

Note that it is hard to find an interesting[11] scheme over $\mathbf{e}$ with $\mathbf{e}$-points $O_{\mathbf{e}}^\times$. So we give up on a nice stack (say, connected and algebraic) $\mathrm{LS}_{\mathbf{G}_m}^{\text{ét}}$ existing.

2.2.3. One the other hand, deformation theory of étale local systems (alias: Galois representations) is an old story. Usually one considers torsion coefficients, but we need not do so here. The basic point is that for an étale local system σ, we have a DG Lie algebra $\mathrm{C}_{\text{ét}}(X, \check{\mathfrak{g}}_\sigma)$, so has an associated formal moduli problem (see [**25**] Chapter 13 and [**18**]).

In other words, although we gave up on $\mathrm{LS}_{\check{G}}^{\text{ét}}$ existing, we *do* know its $\mathbf{e}$-points (which are local systems after all) and we *do* know its formal completion at each such point.

So at the moment, to form *some* approximation to $\mathrm{LS}_{\check{G}}^{\text{ét}}$, we can take a (typically uncountable) disjoint union of the "formal completions of $\mathrm{LS}_{\check{G}}^{\text{ét}}$" at each $\mathbf{e}$-point σ. This gives the right answer for $\mathbf{G}_m$, but we will construct something a little better for other groups (as will be discussed in the remainder of §2.2).

2.2.4. *The definition.* In [**2**], we define a *prestack* over $\mathbf{e}$ (i.e., functor from connective commutative $\mathbf{e}$-algebras to ∞-groupoids, i.e., moduli problem) called $\mathrm{LS}_H^{\mathrm{restr}}$ for any affine algebraic group $H/\mathbf{e}$. In general, it remembers a little more than just formal neighborhoods of points, as we will see.

The definition is a naive imitation of (2.3); by definition, an S-point of $\mathrm{LS}_H^{\mathrm{restr}}$ is a right t-exact symmetric monoidal functor:

$$\mathsf{Rep}(H) \to \mathsf{qLisse}(X) \otimes \mathsf{QCoh}(S).$$

Remark 2.2.4.1. To make this definition appear more concrete, let us explain what the right hand side is without using tensor products of DG categories. Suppose $\mathcal{C}$ is a DG category, which we remind has all colimits. Suppose $S = \mathrm{Spec}(A)$. Then $\mathcal{C} \otimes \mathsf{QCoh}(S) = \mathcal{C} \otimes A\text{-}\mathsf{mod} = A\text{-}\mathsf{mod}(\mathcal{C})$, i.e., an object of $\mathcal{C}$ with an action of A. So the right hand side is reasonably concrete – the complexity is about the same as that for A-mod.

2.2.5. *What do we get?* The above is a formal definition. It remains to justify that we have given a *good notion*, where the meaning of this phrase will become more refined as we proceed.

[11] Here is an example that is not interesting: $\coprod_{O_{\mathbf{e}}^\times} \mathrm{Spec}(\mathbf{e})$. In the case of $\mathbf{G}_m$, $\mathrm{LS}_{\mathbf{G}_m}^{\mathrm{restr}}$ will differ from this answer in stackiness and (possibly cohomological) nilpotents (besides replacing $O_{\mathbf{e}}^\times$ by $(O_{\mathbf{e}}^\times)^{2g}$, of course).

2.2.6. *Example: the additive group.* First, suppose $H = \mathbf{G}_a$.

We claim that in this case, $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_a}$ actually "looks the same" as in the de Rham case! More precisely, we will show that $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_a}$ is the algebraic stack corresponding to the complex $\mathrm{C}_{\text{ét}}(X)[1]$, i.e., it is (non-canonically[12]) isomorphic to $\mathbf{B}H^0_{\text{ét}}(X) \times H^1_{\text{ét}}(X) \times \Omega H^2_{\text{ét}}(X)$ (where $\Omega H^2_{\text{ét}}(X)$ is defined as the derived scheme $0 \times_{H^2_{\text{ét}}(X)} 0$). One can see (e.g., via the following analysis) that the same holds in the de Rham setting, but with de Rham cohomology replacing étale everywhere.

To see this, let $\mathrm{triv} \in \mathsf{Rep}(\mathbf{G}_a)$ be the trivial representation, i.e., the tensor unit. There is a canonical map $\mathrm{triv} \to \mathrm{triv}[1] \in \mathsf{Rep}(\mathbf{G}_a)$ corresponding to the standard 2-dimensional representation $\left(\begin{smallmatrix} 1 & \star \\ 0 & 1 \end{smallmatrix}\right)$ of $\mathbf{G}_a$ (considered as a non-trivial self-extension of the trivial representation).

Now for $\mathcal{C}$ a symmetric monoidal DG category and $F : \mathsf{Rep}(\mathbf{G}_a) \to \mathcal{C}$ a symmetric monoidal functor, we can apply F to the extension class above to obtain a map $\mathbb{1}_{\mathcal{C}} \to \mathbb{1}_{\mathcal{C}}[1]$, i.e., a point in the ($\infty$-)groupoid $\mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}[1]) = \Omega^{\infty-1}\underline{\mathrm{End}}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}})$.[13] It is easy to see[14] that this gives an isomorphism of groupoids:

$$\mathrm{Hom}_{\mathsf{ComAlg}(\mathsf{DGCat}_{\mathrm{cont}})}(\mathsf{Rep}(\mathbf{G}_a), \mathcal{C}) \xrightarrow{\simeq} \mathrm{Hom}_{\mathcal{C}}(\mathbb{1}_{\mathcal{C}}, \mathbb{1}_{\mathcal{C}}[1]).$$

Taking $\mathcal{C} = \mathsf{qLisse}(X) \otimes \mathsf{QCoh}(S)$, we see that S-points of $\mathrm{LS}^{\mathrm{restr}}_H$ equal:

$$\Omega^\infty\big(\underline{\mathrm{Hom}}_{\mathsf{qLisse}(X)}(\mathbf{e}_X, \mathbf{e}_X[1]) \otimes \underline{\mathrm{Hom}}_{\mathsf{QCoh}(S)}(\mathcal{O}_X, \mathcal{O}_X)\big) = \Omega^\infty\big(\,\mathrm{C}_{\text{ét}}(X)[1] \otimes \Gamma(S, \mathcal{O}_S)\big)$$

for $\mathbf{e}_X$ the constant sheaf on X. Up to unwinding the formalism, this proves the claim.

2.2.7. *Example: the multiplicative group.* Here we simply state the outcome:

The space $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_m}$ is an ind-algebraic stack. It is a disjoint union of its connected components, each of which is (again non-canonically[15]) isomorphic to $\mathbf{BG}_m \times H^1_{\text{ét}}(X)^\wedge_0 \times \Omega H^2_{\text{ét}}(X)$. The connected components of $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_m}$ are in bijection with its $\mathbf{e}$-points, which we remind are just the rank 1 lisse sheaves on X.

2.2.8. *What is the toolkit?* This material can be ignored. For the reader's convenience, we describe the general recipes for proving things about $\mathrm{LS}^{\mathrm{restr}}_H$.

First, we need to probe the underlying classical stack, ignoring issues about derived algebraic geometry. For this, we let Π_X be the Tannakian group attached to the Tannakian category $\mathsf{qLisse}(X)^\heartsuit$, so Π_X is a group scheme over $\mathbf{e}$ with a symmetric monoidal equivalence $\mathsf{Rep}(\Pi_X)^\heartsuit \simeq \mathsf{qLisse}(X)^\heartsuit$.

For classical schemes, S-points of $\mathrm{LS}^{\mathrm{restr}}_H$ are canonically in bijection with maps $\Pi_X \times S \to H \times S$ of group schemes over S, considered up to conjugation (where we quotient in the groupoid sense) – see [**2**] Proposition 2.5.9 (though the assertion is essentially Tannakian duality plus bookkeeping). This allows us to study the underlying classical prestack of $\mathrm{LS}^{\mathrm{restr}}_H$ using tools from the theory of algebraic groups.

We then extend to derived schemes using deformation theory, which is simple to compute for $\mathrm{LS}^{\mathrm{restr}}_H$; see [**2**] §2.2.

Example 2.2.8.1. Let us illustrate the first technique in an example.

The earlier assertion that $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_m}$ is a disjoint union of "fat points" from §2.2.7 amounts to saying that for any algebraically closed field extension $\mathbf{e}'/\mathbf{e}$, a map $S = \mathrm{Spec}(\mathbf{e}') \to \mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_m}$ factors through an $\mathbf{e}$-point. This becomes a general assertion about group schemes: a map $\Pi_X \times S \to \mathbf{G}_m \times S$ comes from a map defined over $\mathbf{e}$. As $\mathbf{G}_m$ has finite type, this reduces to the same assertion with Π_X replaced by an affine algebraic group Γ (i.e., a finite type quotient of Π_X), which we can even assume is abelian. Here the assertion is evident from the representation theory of commutative algebraic groups.

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[12] We have in effect chosen a formality isomorphism for $\mathrm{C}_{\text{ét}}(X)[1]$.

[13] To clarify for the reader who is not versed in this material: for a spectrum V (or complex of k-vector spaces), $\Omega^\infty V$ means "take the underlying ∞-groupoid" – at least for connective spectra/chain complexes, this is analogous to taking the underlying set of an abelian group, and in general, one can think of it as "pass to the connective cover and then take the underlying homotopy set." In explicit set-theoretic models, we might take a chain complex $V^\bullet$ of $\mathbf{Z}$-modules, truncate to obtain $\tau^{\leq 0}(V^\bullet)$, and then pass to the corresponding simplicial abelian group (hence simplicial set) under Dold-Kan.

The notation $\Omega^{\infty-1}(V)$ simply means $\Omega^\infty(V[1])$.

[14] Namely, one simply uses that there is a standard symmetric monoidal equivalence between $\mathsf{Rep}(\mathbf{G}_a)$ and modules over the commutative algebra $\mathbf{e} \times \mathbf{e}[-1]$.

[15] Here is a recipe to construct the component more canonically. First, take $\mathrm{LS}^{\mathrm{restr}}_{\mathbf{G}_a}$ and formally complete it at the trivial $\mathbf{G}_a$-local system. The resulting stack receives a homomorphism from $\mathbf{BG}_a^\wedge = \mathbf{BG}_m^\wedge$ (here we use the exponential); then pushout along the map to $\mathbf{BG}_m$.

2.2.9. *Structure of* $\mathrm{LS}^{\mathrm{restr}}$ *in general.* We hope the following results will contain no surprises at this point.

First, $\mathrm{LS}_H^{\mathrm{restr}}$ is always a *formal* algebraic stack. More precisely, if one maps $\mathrm{LS}_H^{\mathrm{restr}} = \mathrm{LS}_H^{\mathrm{restr}}(X) \to \mathbf{B}H = \mathrm{LS}_H^{\mathrm{restr}}(\mathrm{Spec}(k))$ by taking the fiber at a k-point in X, this map is representable in indschemes, and even better, in indschemes that are disjoint unions of formal schemes – see [**2**] Theorem 1.4.5.

Second, the connected components of $\mathrm{LS}_H^{\mathrm{restr}}$ are in bijection with *semi-simple* H-local systems up to equivalence. Informally, two points of $\mathrm{LS}_H^{\mathrm{restr}}$ lie in the same connected component if and only if their semi-simplifications are infinitesimally close. See [**2**] Proposition 3.7.2 for a precise statement.

Finally, if we imagine $\mathrm{LS}_H^{\text{ét}}$ existed, then for each semi-simple σ, there would be a closed substack $\mathrm{LS}_{H,\sigma}^{\text{ét}}$ of local systems with semi-simplification σ; $\mathrm{LS}_H^{\mathrm{restr}}$ is then morally the disjoint union of $\mathrm{LS}_H^{\text{ét}}$ formally completed at each such $\mathrm{LS}_{H,\sigma}^{\text{ét}}$. For more precise assertions in the Betti and de Rham settings, see [**2**] §4.

2.3. Restricted geometric Langlands. We briefly discuss our main conjecture in the subject.

2.3.1. Let $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \subseteq \mathsf{Shv}(\mathrm{Bun}_G)$ denote the subcategory of sheaves with *singular support in the nilpotent cone*. Here singular support of étale sheaves was defined by Beilinson in [**5**].

In the Betti setting, Ben-Zvi and Nadler [**7**] said that sheaves with nilpotent singular support are the right object to study (one finds a precise theorem justifying this idea in [**2**] Theorem 18.1.6). We mimic this principle in the étale setting, conjecturing:

Conjecture 2.3.1 (Restricted geometric Langlands conjecture). *There is an equivalence* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$.

Here the right hand side is defined as in the de Rham case.

Remark 2.3.1.1. One can find a simplified version of this conjecture in [**23**] Conjecture 6.3.2.

Remark 2.3.1.2. Of course, Conjecture 2.3.1 is subject to many compatibilities. The compatibility with Whittaker coefficients, (a mild form of) the compatibility with Eisenstein series,[16] and a version of [**10**] Theorem 8.3.0.1 *uniquely* determine the comparison functor in Conjecture 2.3.1; in the de Rham and Betti settings, this idea is the subject of [**16**].

2.3.2. *Evidence.* When the geometric field k has characteristic 0, we show in [**2**] that the restricted GLC follows from the de Rham geometric Langlands conjecture.

In general, one can directly verify the conjecture for $G = \mathbf{G}_m$, and using similar ideas as in [**21**], one can reduce the conjecture to derived Satake for $X = \mathbf{P}^1$.

2.4. Frobenius. We now discuss what happens when we include Frobenius.

2.4.1. Now suppose the ground field k is $\overline{\mathbf{F}}_q$. Suppose X is defined over $\mathbf{F}_q$; as G is a priori defined over $\mathbf{Z}$, it follows that Bun_G is naturally defined over $\mathbf{F}_q$ as well. Recall that rational structure can be encoded in the geometric Frobenius endomorphism of X (resp. Bun_G).

Therefore, there are Frobenius automorphisms (namely: pullback along geometric Frobenius) acting on $\mathsf{Shv}(X)$, $\mathsf{qLisse}(X)$, and $\mathsf{Shv}(\mathrm{Bun}_G)$.

By definition of $\mathrm{LS}_H^{\mathrm{restr}}$, the Frobenius automorphism of $\mathsf{qLisse}(X)$ induces a "Frobenius" automorphism of $\mathrm{LS}_H^{\mathrm{restr}}$.

Example 2.4.1.1. Suppose $H = \mathbf{G}_a$. By §2.2.6, $\mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}}$ is a geometric avatar of the chain complex $C_{\text{ét}}(X)[1]$; this complex carries its own standard Frobenius automorphism, and the two tautologically match under this dictionary.

In general, the Frobenius on $\mathrm{LS}_H^{\mathrm{restr}}$ might therefore be thought of as a non-linear analogue of the Frobenius on $C_{\text{ét}}(X)$.

[16]See [**13**] for formulations of both Whittaker and Eisenstein compatibilities.

2.4.2. We define $\mathrm{LS}_H^{\mathrm{arthm}}$ as the Frobenius fixed points of $\mathrm{LS}_H^{\mathrm{restr}}$.

More precisely, we have a Cartesian diagram:

$$\begin{tikzcd} \mathrm{LS}_H^{\mathrm{arthm}} \arrow[r] \arrow[d] & \mathrm{LS}_H^{\mathrm{restr}} \arrow[d] \\ \mathrm{LS}_H^{\mathrm{restr}} \arrow[r, "\Delta"] & \mathrm{LS}_H^{\mathrm{restr}} \times \mathrm{LS}_H^{\mathrm{restr}} \end{tikzcd}$$

of formal derived stacks where the arrow on the right is the graph of the Frobenius map.

Note that $\mathbf{e}$-points of $\mathrm{LS}_{GL_n}^{\mathrm{arthm}}$ are rank n lisse Weil sheaves on X. More generally, we think of $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ as the stack parametrizing continuous homomorphisms from the Weil group $\mathcal{W}_X$ to the algebraic group $\check{G}/\mathbf{e}$, considering these homomorphisms up to conjugacy. Said more neatly: $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ is the stack of unramified Langlands parameters for the global field $\mathbf{F}_q(X)$.

Remark 2.4.2.1. Let $\mathsf{qLisse}^{\phi}(X)$ denote the DG category of *quasi-lisse Weil sheaves*, which by definition are the fixed points of the $\mathbf{Z}$-action on $\mathsf{qLisse}(X)$ coming from Frobenius. Tautologically, $\mathrm{LS}_H^{\mathrm{arthm}}$ parametrizes symmetric monoidal functors $\mathsf{Rep}(H) \to \mathsf{qLisse}^{\phi}(X)$ in the same way that $\mathrm{LS}_H^{\mathrm{restr}}$ parametrizes symmetric monoidal functors $\mathsf{Rep}(H) \to \mathsf{qLisse}(X)$ (i.e., S-points of $\mathrm{LS}_H^{\mathrm{arthm}}$ are right t-exact symmetric monoidal functors $\mathsf{Rep}(H) \to \mathsf{qLisse}(X) \otimes \mathsf{QCoh}(S)$).

However, $\mathsf{qLisse}^{\phi}(X)$ has different categorical properties than $\mathsf{qLisse}(X)$. For example, $\mathsf{qLisse}^{\phi}(X)^{\heartsuit}$ is not a Tannakian category. This leads to some formal differences between the two settings, with $\mathrm{LS}_H^{\mathrm{arthm}}$ behaving more like the moduli of Betti local systems in some regards; e.g., it turns out ([**2**] Theorem 16.1.4) that $\mathrm{LS}_H^{\mathrm{arthm}}$ is a (non-formal!) algebraic stack that is quasi-compact (and in particular: has finitely many connected components!).

Remark 2.4.2.2. We do not try to provide more explicit pictures in this section, beyond commenting that the geometry of $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ is more complicated than its restricted counterpart. But in Theorem 4.3.3.1, we give coordinates on a patch of $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ containing the trivial representation, providing some bit of explicit analysis of its geometry.

2.4.3. Essentially by Remark 2.3.1.2, any restricted geometric Langlands equivalence *must* be compatible with Frobenius automorphisms on both sides.

Recall the notion of *categorical trace* alluded to in §2.1.6: it takes (dualizable) DG categories with endofunctors and produces vector spaces.

We can then take the trace of Frobenius on both sides of the restricted geometric Langlands equivalence. As outlined in [**2**] §16, the trace of the Frobenius on $\mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$ is the same as on $\mathsf{IndCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$, which is:

$$\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega)$$

for ω the *dualizing* sheaf on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$.

On the other hand, the main theorem of [**4**] calculates the trace of Frobenius on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ as:

$$\mathcal{A}ut_{G,c}^{\mathrm{unr}}.$$

Using (2.2), one can interpret this as a higher categorical version of the sheaves-functions correspondence (albeit in a special case, not as a general geometric phenomenon).

2.4.4. We end up with the *arithmetic* conjecture:

$$\mathcal{A}ut_{G,c}^{\mathrm{unr}} \simeq \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega).$$

As in the introduction, the vector space on the left is that of *unramified automorphic functions*, i.e., compactly supported functions on $\mathrm{Bun}_G(\mathbf{F}_q)$. In particular, unramified cusp forms sit in this space.

2.4.5. There is a canonical map $\tau : \mathcal{O}_{\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}} \to \omega_{\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}}$ encoding a "weak Calabi-Yau" structure on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ – see §4.6.5 for more discussion.

In particular, there is a natural map from functions on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ to the right hand side $\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega)$ above. One should think of Arthur's SL_2 as measuring the difference between $\mathcal{O}$ and ω on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$.

EXAMPLE 2.4.5.1. Suppose $\sigma \in \mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$ is a smooth, isolated point of this stack. (Such σ are called an *elliptic* or *discrete* Langlands parameter.) Then one can see that $\tau|_\sigma$ is an isomorphism. Therefore, our conjecture predicts that there is a 1-dimensional space of unramified automorphic forms corresponding[17] to σ.

2.5. Xue's theorem and the Frobenius trace.

2.5.1. Above, we said that:

$$\mathrm{tr}(\mathrm{Frob}^*_{\mathrm{Bun}_G}, \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)) \simeq \mathcal{A}ut_c^{\mathrm{unr}} \tag{2.5}$$

was the main theorem of [**4**]. We briefly indicate how this is proved. They key role is played by Xue's theorem from [**31**].

One can also turn to the introduction of [**4**] for an overview of the argument. Our summary is not so different here, except we try a little harder to sweep Beilinson's spectral projector under the rug (maybe to the detriment of the discussion).

2.5.2. *Step 1.* One lesson from Drinfeld's work[18] on the Langlands correspondence is that it is generally helpful to consider automorphic functions $\mathcal{A}ut_c^{\mathrm{unr}}$ as special cases of *sheaves of shtuka cohomologies*.

We briefly review the story. The shtuka construction takes a finite set I, a representation $V \in \mathsf{Rep}(\check{G}^I)$, and yields a sheaf $\mathrm{Sht}_{I,V} \in \mathsf{Shv}(X^I)$. Namely, attached to the data of I and V, one has a *Hecke functor*:

$$H_V : \mathsf{Shv}(\mathrm{Bun}_G) \to \mathsf{Shv}(\mathrm{Bun}_G \times X^I).$$

This functor comes from a naturally defined *kernel* $\mathcal{K}_V \in \mathsf{Shv}(\mathrm{Bun}_G \times \mathrm{Bun}_G \times X^I)$. We remark that geometric Satake plays a key role in the construction, and we refer to [**22**] or [**4**] for more details on the construction.

Then $\mathrm{Sht}_{I,V}$ is obtained by $*$-pulling back $\mathcal{K}_V$ along the graph of Frobenius:

$$\mathrm{Bun}_G \times X^I \xrightarrow{\mathrm{Graph}_{\mathrm{Frob}} \times \mathrm{id}_{X^I}} \mathrm{Bun}_G \times \mathrm{Bun}_G \times X^I$$

and then taking compactly supported cohomology along the Bun_G factor, i.e., !-pushing forward to X^I.

For example, when $I = \varnothing$ (and V is the 1-dimensional representation of the trivial group), $\mathcal{K}_V = \Delta_!(\mathbf{e}_{\mathrm{Bun}_G})$, so by base-change, the above computes $\mathrm{C}_{\text{ét},c}(\mathrm{Bun}_G(\mathbf{F}_q)) = \mathcal{A}ut_G^{\mathrm{unr}}$.

There are natural morphisms between shtuka cohomology sheaves. First, for I fixed, the above construction yields a functor $\mathrm{Sht}_I : \mathsf{Rep}(\check{G}^I) \to \mathsf{Shv}(X^I)$. But we can also vary I; more precisely, the symmetric monoidal structure on $\mathsf{Rep}(\check{G})$ maps the assignment $I \mapsto \mathsf{Rep}(\check{G}^I) = \mathsf{Rep}(\check{G})^{\otimes I}$ into a functor $\mathsf{fSet} \to \mathsf{DGCat}_{\mathrm{cont}}$ (for fSet the category of finite sets); $*$-pullback along diagonal morphisms makes the assignment $I \mapsto \mathsf{Shv}(X^I)$ into a functor $\mathsf{fSet} \to \mathsf{DGCat}_{\mathrm{cont}}$ as well. Then standard functoriality properties of geometric Satake say we have a natural transformation:

$$\mathrm{Sht} : (I \mapsto \mathsf{Rep}(\check{G}^I)) \to (I \mapsto \mathsf{Shv}(X^I))$$

of functors:

$$\mathsf{fSet} \to \mathsf{DGCat}_{\mathrm{cont}}.$$

This functoriality is a key property of shtuka cohomologies, and its existence encodes key symmetries of automorphic functions: V. Lafforgue used exactly this functoriality in [**22**] to construct excursion operators.

[17]We are being sloppy about what "corresponding to" means here. To be more precise, our conjecture combined with the discussion of [**2**] §24.2 implies that there should be a 1-dimensional space of unramified eigenforms for the action of V. Lafforgue's *excursion algebra* with the eigenvalue being that defined by σ. As is well-known, for general G, Hecke operators alone are not enough to pick out a 1-dimensional eigenspace.

[18]The perspective discussed here for general reductive groups is from [**29**].

2.5.3. *Step 2.* We now similarly generalize the other side of our theorem, which we remind is $\mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob})$. The answer should input $V \in \mathsf{Rep}(\check{G}^I)$ and yield a sheaf on X^I, which we will ultimately denote by $\mathrm{Sht}^{\mathrm{tr}}_{I,V}$. Of course, the construction should involve Hecke functors and $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$, so we presently digress to discuss the latter subject for a moment.

Following [**27**] in the topological setting, we show in [**2**] Theorem 14.2.4 (and its subsequent discussion) that for $V \in \mathsf{Rep}(\check{G}^I)$, the Hecke functor H_V maps $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \subseteq \mathsf{Shv}(\mathrm{Bun}_G)$ into $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X^I) \subseteq \mathsf{Shv}(\mathrm{Bun}_G \times X^I)$.

Moreover, we prove a converse as well: in *loc. cit.* Theorem 14.4.3, we show that for $\mathcal{F} \in \mathsf{Shv}(\mathrm{Bun}_G)$ with $H_V(\mathcal{F}) \in \mathsf{Shv}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X)$ for all $V \in \mathsf{Rep}(\check{G})$, one *necessarily has* $\mathcal{F} \in \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$.[19]

So we summarize with the motto: $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ *can be regarded as the subcategory of sheaves* $\mathcal{F}$ *whose Hecke transforms* $H_{V,x}(\mathcal{F})$ *are locally constant as we vary the point* $x \in X$.

This perspective on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ is actually the better one for almost[20] every result in the AGKRRV series. (From one point of view, this is why it is important to introduce Hecke functors and general shtuka sheaves into our present analysis: $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ itself is best understood using the Hecke action.)

2.5.4. *Step 3.* By the above, for $V \in \mathsf{Rep}(\check{G})$, we have a Hecke functor:

$$H_V : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X^I).$$

We can precompose this functor with the Frobenius on Bun_G to obtain:

$$H_V \circ \mathrm{Frob}_{\mathrm{Bun}_G} : \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \to \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \otimes \mathsf{qLisse}(X^I).$$

We can then take the trace along[21] $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$ to obtain an object of $\mathsf{qLisse}(X^I)$. This is the desired object $\mathrm{Sht}^{\mathrm{tr}}_{I,V}$.

Our goal in what follows is to show that we have functorial identifications:

$$\mathrm{Sht}^{\mathrm{tr}}_{I,V} \simeq \mathrm{Sht}_{I,V}[2|I|] \tag{2.6}$$

where the cohomological shift occurs for technical reasons that will appear below. The case $I = \varnothing$, V 1-dimensional now recovers (2.5) in concise notation.

2.5.5. *Step 4.* Observe a difference between $\mathrm{Sht}^{\mathrm{tr}}_{I,V}$ and $\mathrm{Sht}_{I,V}$: for essentially geometric reasons, $\mathrm{Sht}^{\mathrm{tr}}_{I}$ takes values in $\mathsf{qLisse}(X^I) \subseteq \mathsf{Shv}(X^I)$, but this is not apparent for Sht_I itself.

In [**3**], we introduce methods for calculating traces on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$. We refer to *loc. cit.* for details, but the summary answer is that traces can be computed using *general geometric ingredients* (upper-* and lower-! functors) plus a *specific ingredient from (geometric) representation theory.* The latter is Beilinson's spectral projector, whose job (for our purposes) is to take compatible (over I) systems of functors $\mathcal{S}_I : \mathsf{Rep}(\check{G}^I) \to \mathsf{Shv}(X^I)$ and produce a compatible systems $\lambda\mathcal{S}_I : \mathsf{Rep}(\check{G}^I) \to \mathsf{qLisse}(X^I)$.

At an imprecise, top level view, the recipe from [**3**] produces the following answer: the *system* of functors $\mathrm{Sht}^{\mathrm{tr}}_{I}$ is the *best approximation* to the *system* of functors Sht_I that takes values in $\mathsf{qLisse}(X^I)$ rather than $\mathsf{Shv}(X^I)$, i.e., it is $\lambda\,\mathrm{Sht}_I$.

Then Xue's theorem [**31**] says that Sht_I itself takes values in $\mathsf{qLisse}(X^I)$, so $\lambda\,\mathrm{Sht}_I$ coincides with Sht_I itself, so we obtain (2.6).

2.5.6. *Step 5.* The above is morally correct, but we now fix one lie. The discussion that follows can be compared with [**4**] Remark 3.2.6.

The functors Sht_I are compatible under upper-* functors as we vary the finite set I. However, the procedure of applying the spectral projector applies for a system of functors $\mathcal{S}_I$ compatible under upper-! functors. So we need a variant $\mathrm{Sht}^!$ of the shtuka functors that are suitably compatible under upper-! functors.

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[19] Technically, there are minor restrictions on the characteristic of the ground field in this assertion. Recall from §1.4 that we always neglect these small characteristics, and we implicitly assume we are away from these characteristics in our discussion here.

[20] The main exception is the *Künneth formula* from [**2**].

[21] This is analogous to saying that if we have a linear transformation $W_1 \to W_1 \otimes W_2$ with W_1 finite-dimensional, we have a corresponding vector in $W_1^\vee \otimes W_1 \otimes W_2$, and we can pair along the first two factors to obtain a "trace along W_1" that is a vector in W_2.

The relevant functors $\mathrm{Sht}^!_{I,-} : \mathsf{Rep}(\check{G}^I) \to \mathsf{Shv}(X^I)$ are characterized by the formula:

$$\mathrm{C}_c(X^I, \mathrm{Sht}^!_{I,V} \overset{!}{\otimes} \mathcal{F}) = \mathrm{C}_c(\mathrm{Bun}_G \times X^I, (\mathrm{Frob} \times \mathrm{id})^*(\mathcal{K}_V \overset{*}{\otimes} p_3^*(\mathcal{F})) \tag{2.7}$$

for $p_3 : \mathrm{Bun}_G \times \mathrm{Bun}_G \times X^I \to X^I$ the projection.[22]

A priori, the result from [**3**] *actually* implies is that $\{\mathrm{Sht}^{\mathrm{tr}}_I\}_{I \in \mathsf{fSet}}$ is the best approximation to the functors $\{\mathrm{Sht}^!_I\}_{I \in \mathsf{fSet}}$ taking values in the subcategories $\mathsf{qLisse} \subseteq \mathsf{Shv}$.

The logic then proceeds by applying Xue's theorem twice. First, this theorem tells us that $\mathrm{Sht}^!_{I,V} = \mathrm{Sht}_{I,V}[2|I|]$ (where $2|I|$ appears as $2\dim(X^I)$) – namely, we simply substitute $\mathrm{Sht}_{I,V}[2|I|]$ in place of $\mathrm{Sht}^!_{I,V}$ in the left hand side of (2.7), and then we apply the identity $\mathcal{G} \overset{!}{\otimes} \mathcal{F} = \mathcal{G} \overset{*}{\otimes} \mathcal{F}[-2\dim]$ for $\mathcal{G}$ being lisse to (functorially) manipulate the resulting expression into the right hand side of (2.7). In particular, Xue's theorem then implies $\mathrm{Sht}^!_{I,V}$ takes values in qLisse, so the previous paragraph implies $\mathrm{Sht}^{\mathrm{tr}}_I = \mathrm{Sht}^!_I$, which we just saw also equals $\mathrm{Sht}_I[2|I|]$, as desired.

3. Spectral Eisenstein series

We now begin working toward Theorem A. For the remainder of the paper, we assume $k = \overline{\mathbf{F}}_q$.

Our goal in this section is to define and study a certain map:

$$\mathrm{Eis}^{\mathrm{spec}} : \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}).$$

Throughout this section, we only consider (pre)stacks locally almost of finite type; we omit further mention of this hypothesis.

3.1. Automorphic Eisenstein series. We begin by reviewing some constructions regarding geometric Eisenstein series and their function-theoretic counterpart, the pseudo-Eisenstein series. We will later wish to find counterparts of these constructions on the spectral side.

3.1.1. First, we have a canonical functor:

$$\mathrm{Eis}_! : \mathsf{Shv}(\mathrm{Bun}_T) \to \mathsf{Shv}(\mathrm{Bun}_G)$$

defined by $*$-pullback along $\mathrm{Bun}_B \to \mathrm{Bun}_T$ followed by !-pushforward along $\mathrm{Bun}_B \to \mathrm{Bun}_G$.

By the Hecke property for $\mathrm{Eis}_!$ established in [**6**], and [**2**] Theorem 14.4.3 (the "converse to the Nadler-Yun theorem," cf. §2.5.3) , we find:

PROPOSITION 3.1.1.1. *The functor* $\mathrm{Eis}_!$ *maps* $\mathsf{qLisse}(\mathrm{Bun}_T)(= \mathsf{Shv}_{\mathcal{N}ilp_T}(\mathrm{Bun}_T))$ *to* $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$.

3.1.2. *Pseudo-Eisenstein series.* Let ps-Eis : $\mathcal{A}ut^{\mathrm{unr}}_{T,c} \to \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ be the *pseudo-Eisenstein* map. By definition, this is the composition:

$$\mathcal{A}ut^{\mathrm{unr}}_{T,c} := \mathrm{Fun}_c(\mathrm{Bun}_T(\mathbf{F}_q)) \to \mathrm{Fun}_c(\mathrm{Bun}_B(\mathbf{F}_q)) \to \mathrm{Fun}_c(\mathrm{Bun}_G(\mathbf{F}_q)) =: \mathcal{A}ut^{\mathrm{unr}}_{G,c}$$

given by first restricting (noting that the fibers of the map $\mathrm{Bun}_B(\mathbf{F}_q) \to \mathrm{Bun}_T(\mathbf{F}_q)$ are finite) and then summing along the fibers of the map $\mathrm{Bun}_B(\mathbf{F}_q) \to \mathrm{Bun}_G(\mathbf{F}_q)$ (which is well-defined because we consider this on functions with finite support).

3.1.3. *Compatibility of the two.* The functor $\mathrm{Eis}_!$ obviously intertwines Frobenii and preserves compact objects, so we may pass to traces of Frobenius to obtain a map:

$$\mathrm{tr}(\mathrm{Eis}_!) : \mathrm{tr}_{\mathsf{qLisse}(\mathrm{Bun}_T)}(\mathrm{Frob}) \to \mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}).$$

By the main theorem of [**4**], we have isomorphisms:

$$\begin{aligned} \mathrm{tr}_{\mathsf{qLisse}(\mathrm{Bun}_T)}(\mathrm{Frob}) &\simeq \mathcal{A}ut^{\mathrm{unr}}_{T,c} \\ \mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}) &\simeq \mathcal{A}ut^{\mathrm{unr}}_{G,c} \end{aligned} \tag{3.1}$$

[22] A notational remark: our notation is inconsistent with [**4**]. The collection of functors we now call $\mathrm{Sht}^!_I$ are neatly packaged in the single functor called Sht in [**4**], although the functors we call Sht_I here are denoted in the same way in [**4**]. They differ only by shifts by Xue's theorem.

so $\mathrm{tr}(\mathrm{Eis}_!)$ corresponds to a map:

$$\mathcal{A}ut_{T,c}^{\mathrm{unr}} \to \mathcal{A}ut_{G,c}^{\mathrm{unr}}.$$

By[23] [**4**] Theorem 5.2.3, the isomorphisms (3.1) are given by a version of the sheaves-functions correspondence; it follows formally that we have a commutative diagram:

$$\begin{tikzcd} \mathrm{tr}_{\mathsf{qLisse}(\mathrm{Bun}_T)}(\mathrm{Frob}) \arrow[r, "\mathrm{tr}(\mathrm{Eis}_!)"] \arrow[d, "\simeq"] & \mathrm{tr}_{\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)}(\mathrm{Frob}) \arrow[d, "\simeq"] \\ \mathcal{A}ut_{T,c}^{\mathrm{unr}} \arrow[r, "\text{ps-Eis}"] & \mathcal{A}ut_{G,c}^{\mathrm{unr}}. \end{tikzcd} \tag{3.2}$$

In other words: *the trace of geometric Eisenstein series is the pseudo-Eisenstein series.*

3.2. Some general constructions. Until further notice, we work exclusively over the field $\mathbf{e}$.

Below, we give general a construction of $\mathrm{Eis}^{\mathrm{spec}}$ in a general stack-theoretic context.

3.2.1. *Terminology around stacks.* Recall the technical notion of a *QCA stack* from [**8**]: this term refers to an algebraic stack $\mathcal{Y}$ that is quasi-compact with affine diagonal. For any such QCA stack $\mathcal{Y}$, [**8**] Theorem 0.4.5 asserts that $\mathsf{IndCoh}(\mathcal{Y}) = \mathsf{Ind}(\mathsf{Coh}(\mathcal{Y}))$. Moreover, by [**8**] §3, there is a good theory of pushforwards for ind-coherent sheaves on QCA stacks.

Also, we recall the notion of *ind-algebraic stack* from [**2**] §5.2. We remind that a prestack $\mathcal{Y}$ is ind-algebraic if it is convergent and for every $n \geqslant 0$ its n-truncation ${}^{\leqslant n}\mathcal{Y}$ can be written as a filtered colimit of n-truncated algebraic stacks $\mathcal{Y}_i$ under closed embeddings. We say $\mathcal{Y}$ is *ind-QCA* if the terms $\mathcal{Y}_i$ can moreover be taken to be QCA.

Our main example is $\mathrm{LS}_H^{\mathrm{restr}}$ for H an affine algebraic group. According to [**2**] Corollary 5.2.6, $\mathrm{LS}_H^{\mathrm{restr}}$ is ind-algebraic; moreover, the proof of this result shows that $\mathrm{LS}_H^{\mathrm{restr}}$ is in fact ind-QCA.

By the above theorem of Drinfeld-Gaitsgory, any ind-QCA stack $\mathcal{Y}$ has $\mathsf{IndCoh}(\mathcal{Y})$ being compactly generated. Again, there is a good theory of IndCoh-pushforwards for morphisms between ind-QCA stacks.

3.2.2. Below, we fix $f : \mathcal{Y} \to \mathcal{Z}$ a 1-representable[24] map between ind-QCA algebraic stacks.

Suppose in addition that we are given automorphisms $\phi_{\mathcal{Y}} : \mathcal{Y} \to \mathcal{Y}$ and $\phi_{\mathcal{Z}} : \mathcal{Z} \to \mathcal{Z}$ intertwined by f (i.e., we are given an identification $\phi_{\mathcal{Z}} \circ f \simeq f \circ \phi_{\mathcal{Y}}$). We sometimes omit the subscripts and simply write ϕ for either $\phi_{\mathcal{Y}}$ or $\phi_{\mathcal{Z}}$.

We form the fixed point stack $\mathcal{Y}^{\phi}$ (resp. $\mathcal{Z}^{\phi}$) of ϕ. Explicitly, this is the equalizer $\mathrm{Eq}(\mathcal{Y} \underset{\mathrm{id}}{\overset{\phi}{\rightrightarrows}} \mathcal{Y})$, which can also be written as the Cartesian product of $\mathcal{Y} \xrightarrow{\mathrm{Graph}_{\phi}} \mathcal{Y} \times \mathcal{Y} \xleftarrow{\Delta} \mathcal{Y}$. By assumption, we have an induced map $\mathcal{Y}^{\phi} \to \mathcal{Z}^{\phi}$ that we denote by f^{ϕ}.

Below, under suitable hypotheses, we will construct canonical maps between $\Gamma(\mathcal{Y}^{\phi}, \omega_{\mathcal{Y}^{\phi}})$ and $\Gamma(\mathcal{Z}^{\phi}, \omega_{\mathcal{Z}^{\phi}})$.

We have structured the discussion as follows. In §3.2.3 and §3.2.4, we have given "elementary" constructions of these maps using standard functoriality properties of IndCoh. The latter construction in particular is somewhat involved. Afterward, we explain a conceptual framework (*functoriality of traces*) for these constructions that makes their existence obvious. The author thinks about these maps using the latter point of view, but fears the reader will not trust the magic if concrete descriptions are lacking; the reader who does not need such convincing can skip past §3.2.3-3.2.4.

We also remark that the ind-QCA assumption is overkill. It is not needed in §3.2.3. It is used mildly in §3.2.4 for the existence of various pushforward functors, though weaker hypotheses suffice. Fundamentally, this hypothesis is natural from the more conceptual perspective of traces to ensure compact generation (hence dualizability) of IndCoh.

[23]We remark that *loc. cit.* is conditional (even in its formulation) on a certain technical hypothesis on $\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G)$; see [**4**] §5.1. This hypothesis was recently verified by the author and Gaitsgory and will appear in forthcoming work.

[24]I.e., the fibers are algebraic stacks. Specifically, for every $S \in \mathsf{AffSch}$ and $S \to \mathcal{Z}$, the fiber product $\mathcal{Y} \times_{\mathcal{Z}} S$ is an algebraic stack. In our context, this condition rules out a map like $\mathcal{Y} \to \mathrm{Spec}(\mathbf{e})$ unless $\mathcal{Y}$ is an actual (as opposed to ind-)algebraic stack.

3.2.3. *Pushforward.* First, suppose that the map f is representable and *proper*. In this case, we will construct a map:

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}) \to \Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^\phi, \omega_{\mathcal{Z}^\phi}). \tag{3.3}$$

In fact, this is quite easy. In this case, the map $f^\phi : \mathcal{Y}^\phi \to \mathcal{Z}^\phi$ is also proper (and representable),[25] which is all we will need below.

Then $f_*^{\phi,\mathsf{IndCoh}}$ is left adjoint to $f^{\phi,!}$, so we obtain a canonical adjunction map $f_*^{\phi,\mathsf{IndCoh}} f^{\phi,!} \to \mathrm{id}$. This yields a map $f_*^{\phi,\mathsf{IndCoh}}(\omega_{\mathcal{Y}^\phi}) \to \omega_{\mathcal{Z}^\phi}$; applying $\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^\phi, -)$ gives the desired map (3.3).

Remark 3.2.3.1. The map (3.3) is $\Gamma(\mathcal{Z}^\phi, \mathcal{O}_{\mathcal{Z}^\phi})$-linear.

3.2.4. *Pullback.* Next, suppose that f is eventually coconnective (and 1-representable). In this case, we will construct a map:

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^\phi, \omega_{\mathcal{Z}^\phi}) \to \Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}). \tag{3.4}$$

First, note[26] that $f_*^{\mathsf{IndCoh}} : \mathsf{IndCoh}(\mathcal{Y}) \to \mathsf{IndCoh}(\mathcal{Z})$ admits a left adjoint $f^{*,\mathsf{IndCoh}}$ in this case. We have base-change between $*$-pushforwards and $*$-pullbacks (with the latter being only considered for eventually coconnective morphisms).

In this case, we have a natural transformation:

$$(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}(\mathrm{id} \times \phi_{\mathcal{Z}})_*^{\mathsf{IndCoh}}(\mathrm{id} \times f)_*^{\mathsf{IndCoh}} =$$
$$(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}(\mathrm{id} \times f)_*^{\mathsf{IndCoh}}(\mathrm{id} \times \phi_{\mathcal{Y}})_*^{\mathsf{IndCoh}} \to (\mathrm{id} \times \phi_{\mathcal{Y}})_*^{\mathsf{IndCoh}} \in \mathsf{End}(\mathsf{IndCoh}(\mathcal{Y} \times \mathcal{Y})).$$

coming from adjunction. Applying this map to $\Delta_*^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}})$, we obtain a canonical map:

$$(\mathrm{id} \times f)^{*,\mathsf{IndCoh}} \mathrm{Graph}_{\phi_{\mathcal{Z}} \circ f,*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}}) \to \mathrm{Graph}_{\phi_{\mathcal{Y}},*}(\omega_{\mathcal{Y}}) \in \mathsf{IndCoh}(\mathcal{Y} \times \mathcal{Y}). \tag{3.5}$$

Here for a map $g : S \to T$, the map $\mathrm{Graph}_g : S \to S \times T$ is the graph morphism, i.e., $\mathrm{Graph}_g := (\mathrm{id} \times g) \circ \Delta_S$.

Let $\varpi_{\mathcal{Y}}$ denote the canonical map $\mathcal{Y}^\phi \to \mathcal{Y}$ sending a pair $(y \in \mathcal{Y}, y \simeq \phi_{\mathcal{Z}}(y))$ to y, and similarly for $\varpi_{\mathcal{Z}}$. Below, we will construct a canonical isomorphism:

$$\Delta_{\mathcal{Y}}^!(\mathrm{id} \times f)^{*,\mathsf{IndCoh}} \mathrm{Graph}_{\phi_{\mathcal{Z}} \circ f,*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}}) \simeq f^{*,\mathsf{IndCoh}} \varpi_{\mathcal{Z},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Z}^\phi}) \in \mathsf{IndCoh}(\mathcal{Y}). \tag{3.6}$$

Assuming for a moment that this construction has been given, we obtain a canonical map:

$$f^{*,\mathsf{IndCoh}} \varpi_{\mathcal{Z},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Z}^\phi}) \to \Delta_{\mathcal{Y}}^! \mathrm{Graph}_{\phi_{\mathcal{Y}},*}(\omega_{\mathcal{Y}}) \in \mathsf{IndCoh}(\mathcal{Y})$$

by applying $\Delta_{\mathcal{Y}}^!$ to (3.5). By adjunction, this yields a canonical map:

$$\varpi_{\mathcal{Z},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Z}^\phi}) \to f_*^{\mathsf{IndCoh}} \Delta_{\mathcal{Y}}^! \mathrm{Graph}_{\phi_{\mathcal{Y}},*}(\omega_{\mathcal{Y}}) \simeq f_*^{\mathsf{IndCoh}} \varpi_{\mathcal{Y},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}^\phi}) \in \mathsf{IndCoh}(\mathcal{Z}).$$

Here we have used the base-change isomorphism $\Delta_{\mathcal{Y}}^! \mathrm{Graph}_{\phi_{\mathcal{Y}},*}^{\mathsf{IndCoh}} \simeq \varpi_{\mathcal{Y},*}^{\mathsf{IndCoh}} \varpi_{\mathcal{Y}}^!$. Now applying the global sections functor $\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}, -)$ to both sides above, we obtain the desired map (3.4).

It remains to give the isomorphism (3.6). First, the Cartesian diagram:

$$\begin{tikzcd}
\mathcal{Y} \arrow[r, "\mathrm{Graph}_{\phi_{\mathcal{Z}} \circ f}"] \arrow[d, "f"] & \mathcal{Y} \times \mathcal{Z} \arrow[d, "f \times \mathrm{id}"] \\
\mathcal{Z} \arrow[r, "\mathrm{Graph}_{\phi_{\mathcal{Z}}}"] & \mathcal{Z} \times \mathcal{Z}
\end{tikzcd}$$

gives a base-change isomorphism:

$$\mathrm{Graph}_{\phi_{\mathcal{Z}} \circ f,*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}}) \simeq (f \times \mathrm{id})^! \mathrm{Graph}_{\phi_{\mathcal{Z}},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Z}}).$$

This now yields:

$$\Delta_{\mathcal{Y}}^!(\mathrm{id} \times f)^{*,\mathsf{IndCoh}} \mathrm{Graph}_{\phi_{\mathcal{Z}} \circ f,*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Y}}) \simeq \Delta_{\mathcal{Y}}^!(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}(f \times \mathrm{id})^! \mathrm{Graph}_{\phi_{\mathcal{Z}},*}^{\mathsf{IndCoh}}(\omega_{\mathcal{Z}}).$$

[25] Indeed, because f is representable and separated, the morphism $\mathcal{Y}^\phi = \mathcal{Y} \times_{\mathcal{Y} \times \mathcal{Y}} \mathcal{Y} \to \mathcal{Y} \times_{\mathcal{Z} \times \mathcal{Z}} \mathcal{Y}$ is a closed embedding. Clearly the further projection $\mathcal{Y} \times_{\mathcal{Z} \times \mathcal{Z}} \mathcal{Y} \to \mathcal{Z} \times_{\mathcal{Z} \times \mathcal{Z}} \mathcal{Z} = \mathcal{Z}^\phi$ is proper.

[26] This, and other similar assertions in this section, formally reduce to the results of [**17**] Chapter 4 §3.

We have[27] $(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}(f \times \mathrm{id})^! \simeq (f \times \mathrm{id})^!(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}$, so we can rewrite the right hand side above as:

$$\Delta^!_{\mathcal{Y}}(f \times \mathrm{id})^!(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}\,\mathrm{Graph}^{\mathsf{IndCoh}}_{\phi_{\mathcal{Z}},*}(\omega_{\mathcal{Z}}) = \mathrm{Graph}^{\sigma,!}_f(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}\,\mathrm{Graph}^{\mathsf{IndCoh}}_{\phi_{\mathcal{Z}},*}(\omega_{\mathcal{Z}}).$$

Here we use Graph^σ_g to denote $(g \times \mathrm{id}) \circ \Delta$, i.e., the graph map following by swapping the two Cartesian factors.

Now base-change for the Cartesian diagrams:

$$\begin{tikzcd} \mathcal{Y} \arrow[r, "\mathrm{Graph}^\sigma_f"] \arrow[d, "f"] & \mathcal{Z} \times \mathcal{Y} \arrow[d, "\mathrm{id} \times f"] \\ \mathcal{Z} \arrow[r, "\Delta_{\mathcal{Z}}"] & \mathcal{Z} \times \mathcal{Z} \end{tikzcd} \qquad \begin{tikzcd} \mathcal{Z}^\phi \arrow[r, "\varpi_{\mathcal{Z}}"] \arrow[d, "\varpi_{\mathcal{Z}}"] & \mathcal{Z} \arrow[d, "\mathrm{Graph}_{\phi_{\mathcal{Z}}}"] \\ \mathcal{Z} \arrow[r, "\Delta_{\mathcal{Z}}"] & \mathcal{Z} \times \mathcal{Z} \end{tikzcd}$$

yields identifications:

$$\mathrm{Graph}^{\sigma,!}_f(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}\,\mathrm{Graph}^{\mathsf{IndCoh}}_{\phi_{\mathcal{Z}},*}(\omega_{\mathcal{Z}}) \simeq f^{*,\mathsf{IndCoh}}\Delta^!_{\mathcal{Z}}\,\mathrm{Graph}^{\mathsf{IndCoh}}_{\phi_{\mathcal{Z}},*}(\omega_{\mathcal{Z}}) \simeq f^{*,\mathsf{IndCoh}}\varpi^{\mathsf{IndCoh}}_{\mathcal{Z},*}(\omega_{\mathcal{Z}^\phi})$$

as desired.

REMARK 3.2.4.1. The map (3.4) is $\Gamma(\mathcal{Z}^\phi, \mathcal{O}_{\mathcal{Z}^\phi})$-linear.

3.2.5. *Categorical setting.* We now present a more conceptual approach to constructions such as the above.

Suppose $F : \mathcal{C} \to \mathcal{D} \in \mathsf{DGCat}_{\mathrm{cont}}$ is a map between dualizable DG categories. Suppose that $\mathcal{C}$ (resp. $\mathcal{D}$) is equipped with an endofunctor $\phi_{\mathcal{C}}$ (resp. $\phi_{\mathcal{D}}$) and that:

- F admits a continuous right adjoint F^R.
- F lax intertwines ϕ, i.e., we are given a map (often an isomorphism) $F \circ \phi_{\mathcal{C}} \to \phi_{\mathcal{D}} \circ F$.

Then standard functoriality of traces yields a canonical map:

$$\mathrm{tr}_{\mathcal{C}}(\phi_{\mathcal{C}}) \to \mathrm{tr}_{\mathcal{D}}(\phi_{\mathcal{D}}) \in \mathsf{Vect}$$

associated with this data. Namely, we have:

$$\mathrm{tr}_{\mathcal{C}}(\phi_{\mathcal{C}}) \to \mathrm{tr}_{\mathcal{C}}(\phi_{\mathcal{C}}F^RF) \simeq \mathrm{tr}_{\mathcal{D}}(F\phi_{\mathcal{C}}F^R) \to \mathrm{tr}_{\mathcal{D}}(\phi_{\mathcal{D}}FF^R) \to \mathrm{tr}_{\mathcal{D}}(\phi_{\mathcal{D}}).$$

Here we used the cyclicity of traces and standard adjunction maps.

3.2.6. *Pushforward/pullback revisted.* Suppose first that we are given $f : \mathcal{Y} \to \mathcal{Z}$ as before representable and proper. We also suppose $\mathcal{Y}$ and $\mathcal{Z}$ are ind-QCA (to ensure IndCoh is dualizable).

In the setting of §3.2.5, take $\mathcal{C} = \mathsf{IndCoh}(\mathcal{Y})$, $\mathcal{D} = \mathsf{IndCoh}(\mathcal{Z})$, $F = f^{\mathsf{IndCoh}}_*$, $\phi_{\mathcal{C}} = \phi^{\mathsf{IndCoh}}_{\mathcal{Y},*}$ and $\phi_{\mathcal{D}} = \phi^{\mathsf{IndCoh}}_{\mathcal{Z},*}$. Then the categorical formalism yields a canonical map:[28]

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}) = \mathrm{tr}_{\mathsf{IndCoh}(\mathcal{Y})}(\phi^{\mathsf{IndCoh}}_{\mathcal{Y},*}) \to \mathrm{tr}_{\mathsf{IndCoh}(\mathcal{Z})}(\phi^{\mathsf{IndCoh}}_{\mathcal{Z},*}) = \Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^\phi, \omega_{\mathcal{Z}^\phi}).$$

A straightforward diagram chase shows that this map recovers (3.3). We remark that properness is needed for f^{IndCoh}_* to admit a continuous right adjoint.

Next, take $f : \mathcal{Y} \to \mathcal{Z}$ 1-representable and eventually coconnective. Now take $\mathcal{C} = \mathsf{IndCoh}(\mathcal{Z})$, $\mathcal{D} = \mathsf{IndCoh}(\mathcal{Y})$, $F = f^{\mathsf{IndCoh},*}$, $\phi_{\mathcal{C}} = \phi^!_{\mathcal{Z}}$, $\phi_{\mathcal{D}} = \phi^!_{\mathcal{Y}}$. Then the categorical formalism yields a canonical map:[29]

$$\Gamma^{\mathsf{IndCoh}}(\mathcal{Z}^\phi, \omega_{\mathcal{Z}^\phi}) = \mathrm{tr}_{\mathsf{IndCoh}(\mathcal{Z})}(\phi^!_{\mathcal{Z}}) \to \mathrm{tr}_{\mathsf{IndCoh}(\mathcal{Y})}(\phi^!_{\mathcal{Y}}) = \Gamma^{\mathsf{IndCoh}}(\mathcal{Y}^\phi, \omega_{\mathcal{Y}^\phi}).$$

A diagram chase shows that this map coincides with (3.4).

REMARK 3.2.6.1. The second diagram chase is routine, but unsurprisingly, somewhat more involved than the first. We omit the verification here. Actually, for our purposes, the reader may take the categorical constructions as *definitions* of (3.3) and (3.4), completely ignoring the material of §3.2.3-3.2.4. We only included the explicit constructions to make the construction appear more concrete.

[27] This kind of commutation is a general fact: see [**12**] Proposition 7.1.6. However, it is particularly easy in the present setting: by the Künneth formula, we can write $(f \times id)^! = f^! \otimes \mathrm{id}$ and $(\mathrm{id} \times f)^{*,\mathsf{IndCoh}}$ as $\mathrm{id} \otimes f^{*,\mathsf{IndCoh}}$.

[28] Here the equalities are standard isomorphisms; see e.g. [**15**] §3.5.3.

[29] Note that $\mathrm{tr}_{\mathcal{C}}(F) = \mathrm{tr}_{\mathcal{C}^\vee}(F^\vee)$, so $\mathrm{tr}(\phi^!) = \mathrm{tr}(\phi^{\mathsf{IndCoh}}_*)$.

3.3. Spectral Eisenstein series.

3.3.1. *Setup.* We have the standard correspondence:

$$\begin{tikzcd} & \mathrm{LS}^{\mathrm{restr}}_{\check{B}} \arrow[dl, "\mathfrak{q}"'] \arrow[dr, "\mathfrak{p}"] & \\ \mathrm{LS}^{\mathrm{restr}}_{\check{T}} & & \mathrm{LS}^{\mathrm{restr}}_{\check{G}}. \end{tikzcd}$$

The map $\mathfrak{p}$ is representable and proper, while the map $\mathfrak{q}$ is quasi-smooth and 1-representable. Also, each of these spaces carries a Frobenius self-map, and the maps $\mathfrak{p}$ and $\mathfrak{q}$ intertwine these Frobenii. Therefore, by (3.3) and (3.4), we obtain canonical maps:

$$\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{B}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{B}}}) \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}).$$

Definition 3.3.1.1. The composition of the above maps is the *spectral Eisenstein series* $\mathrm{Eis}^{\mathrm{spec}} : \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}})$.

3.4. Spectral vs. function theoretic Eisenstein series.

3.4.1. Recall that *restricted geometric Langlands* (see Conjecture 2.3.1) predicts an equivalence of categories:

$$\mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \simeq \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}). \tag{3.7}$$

The equivalence should be subject to various compatibilities. We highlight two of salient interest here:

- (Hecke compatibility): The equivalence (3.7) is of $\mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}})$-module categories; here the right hand side has the evident action and the left hand side carries the action of [**2**] Theorem 14.3.2.
- (Eisenstein compatibility, $P = B$ case): The diagram:

$$\begin{tikzcd} \mathsf{qLisse}(\mathrm{Bun}_T) \arrow[r, "\mathrm{Eis}_!"] \arrow[d] & \mathsf{Shv}_{\mathcal{N}ilp}(\mathrm{Bun}_G) \arrow[d, "\simeq"] \\ \mathsf{QCoh}(\mathrm{LS}^{\mathrm{restr}}_{\check{T}}) \arrow[r, "\mathrm{Eis}^{\mathrm{spec}}"] & \mathsf{IndCoh}_{\mathcal{N}ilp^{\mathrm{spec}}}(\mathrm{LS}^{\mathrm{restr}}_{\check{G}}) \end{tikzcd}$$

  commutes; here the left arrow is the equivalence unconditionally constructed in [**2**] Example 21.2.9.

3.4.2. We now recall the following result:

Proposition 3.4.2.1. *Suppose $\mathcal{Y}$ is a quasi-smooth ind-algebraic stack equipped with a self-map $\phi_{\mathcal{Y}}$. Let $\mathcal{N} \subseteq T^*[-1]\mathcal{Y}$ be a closed conical substack (of the -1-shifted cotangent bundle of $\mathcal{Y}$) such that for every point $y \in Y^{\phi}$, the map $d\phi[-1] : \mathcal{N}_y \to \mathcal{N}_y$ is contracting onto $0 \in T^*_y[-1]\mathcal{Y}$. Then the map:*

$$\mathrm{tr}_{\mathsf{IndCoh}_{\mathcal{N}}(\mathcal{Y})}(\phi^!) \to \mathrm{tr}_{\mathsf{IndCoh}(\mathcal{Y})}(\phi^!) = \Gamma(\mathcal{Y}^{\phi}, \omega_{\mathcal{Y}^{\phi}})$$

is an isomorphism.

See [**2**] §24.6.8.[30]

In particular, we find that restricted geometric Langlands produces an isomorphism:

$$\mathcal{A}ut^{\mathrm{unr}}_{G,c} \simeq \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}). \tag{3.8}$$

Assuming the Hecke compatibility for restricted geometric Langlands, this is an equivalence of $\mathcal{E}xc_{\check{G}} := \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O}_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}})$-modules, where the left hand side inherits its $\mathcal{E}xc_{\check{G}}$-module structure from [**4**]. We remind (see [**2**] §24.2) that the $\mathcal{E}xc_{\check{G}}$-module structure on $\mathcal{A}ut^{\mathrm{unr}}_c$ refines the Lafforgue-Xue action of excursion operators on this space (see §1.2.3).

[30] In *loc. cit.*, this is formulated as a conjecture. But it is actually straightforward to prove from the formalism of [**1**].

3.4.3. We now obtain:

Proposition 3.4.3.1. *Assume restricted geometric Langlands holds for G with its Eisenstein compatibility. Then the following diagram commutes:*

$$\begin{tikzcd} \mathcal{A}ut^{\mathrm{unr}}_{T,c} \arrow[r, "\text{ps-Eis}"] \arrow[d, "\simeq"] & \mathcal{A}ut^{\mathrm{unr}}_{G,c} \arrow[d, "\simeq"] \\ \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \arrow[r, "\mathrm{Eis}^{\mathrm{spec}}"] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}) \end{tikzcd}$$

where the vertical isomorphisms come from (3.8)*.*

Indeed, this follows from the realizations of ps-Eis and $\mathrm{Eis}^{\mathrm{spec}}$ via traces, see (3.2) and §3.2.6.

3.5. Formulation of the main result.

3.5.1. We will be concerned with *localized* versions of the map $\mathrm{Eis}^{\mathrm{spec}}$. We briefly discuss the relevant formalism.

Suppose we are given a commutative diagram:

$$\begin{tikzcd} & \mathrm{LS}^{\mathrm{arthm}}_{\check{B}} \arrow[dl] \arrow[dr] & \\ \mathrm{LS}^{\mathrm{arthm}}_{\check{G}} \arrow[dr, "f"'] & & \mathrm{LS}^{\mathrm{arthm}}_{\check{T}} \arrow[dl, "g"] \\ & \mathbf{A}^1 & \end{tikzcd}$$

Let $\mathring{\mathbf{A}}^1 := \mathbf{A}^1 \backslash 0$. Note that:

$$\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega)[f^{-1}] := \operatorname*{colim}_{f\cdot -} \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}) \simeq \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}} \underset{\mathbf{A}^1}{\times} \mathring{\mathbf{A}}^1, \omega)$$

and similarly for $(\check{T}, g)$ or $(\check{B}, f|_{\mathrm{LS}_{\check{B}^{\mathrm{arthm}}}} = g|_{\mathrm{LS}_{\check{B}^{\mathrm{arthm}}}})$ in place of $(\check{G}, f)$.

Now observe that we have a correspondence:

$$\begin{tikzcd} & \mathrm{LS}^{\mathrm{arthm}}_{\check{B}} \underset{\mathbf{A}^1}{\times} \mathring{\mathbf{A}}^1 \arrow[dl] \arrow[dr] & \\ \mathrm{LS}^{\mathrm{arthm}}_{\check{G}} \underset{\mathbf{A}^1}{\times} \mathring{\mathbf{A}}^1 & & \mathrm{LS}^{\mathrm{arthm}}_{\check{T}} \underset{\mathbf{A}^1}{\times} \mathring{\mathbf{A}}^1 \end{tikzcd} \tag{3.9}$$

with left arrow proper and representable and right arrow eventually coconnective and 1-representable. As in the definition of spectral Eisenstein series, we obtain a canonical map:

$$\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}})[g^{-1}] \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}})[f^{-1}]. \tag{3.10}$$

We clearly have:

Lemma 3.5.1.1. *(1) The map $\mathrm{Eis}^{\mathrm{spec}} : \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \to \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}})$ intertwines the operators of multiplication by f and g, i.e., the map is naturally a morphism of $\mathbf{e}[t]$-modules.*

(2) The map (3.10) *is obtained by inverting the action of t, i.e., tensoring over $\mathbf{e}[t]$ with $\mathbf{e}[t, t^{-1}]$.*

(3) The diagram:

$$\begin{tikzcd} \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}}) \arrow[r, "\mathrm{Eis}^{\mathrm{spec}}"] \arrow[d] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}}) \arrow[d] \\ \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{T}}})[g^{-1}] \arrow[r, "(3.10)"] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \omega_{\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}})[f^{-1}] \end{tikzcd}$$

commutes.

Accordingly, when the meaning is unambiguous, we will abuse notation in also denoting the map (3.10) by $\mathrm{Eis}^{\mathrm{spec}}$.

3.5.2. We are now in a position to state our main result about $\mathrm{Eis}^{\mathrm{spec}}$.

THEOREM 3.5.2.1. *There exist functions $\delta_{\check{G}} \in \Omega^\infty\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \mathcal{O})$ and $\delta_{\check{T}} \in \Omega^\infty\Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \mathcal{O})$ fitting into a commutative diagram:*

$$\begin{tikzcd} & \mathrm{LS}_{\check{B}}^{\mathrm{arthm}} \arrow[dl] \arrow[dr] & \\ \mathrm{LS}_{\check{G}}^{\mathrm{arthm}} \arrow[dr, "\delta_{\check{G}}"'] & & \mathrm{LS}_{\check{T}}^{\mathrm{arthm}} \arrow[dl, "\delta_{\check{T}}"] \\ & \mathbf{A}^1 & \end{tikzcd} \tag{3.11}$$

such that:

1. *The map $\delta_{\check{G}}$ takes a non-zero value at the trivial Weil local system.*
2. *The object:*
$$\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega_{\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}})[\delta_{\check{G}}^{-1}] \in \mathsf{Vect}$$
lies in cohomological degree 0,[31] *and similarly with $\check{T}$ replacing $\check{G}$.*
3. *The map:*
$$\mathrm{Eis}^{\mathrm{spec}} : \Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \omega_{\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}})[\delta_{\check{T}}^{-1}] \to \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega_{\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}})[\delta_{\check{G}}^{-1}] \in \mathsf{Vect}^{\heartsuit} \tag{3.12}$$
is surjective.

This result will be proved in §4.

3.6. Proof of Theorem A. We now deduce the main theorem of this paper from Theorem 3.5.2.1 and our earlier observations. We remind that we have assumed G is semi-simple[32] here.

3.6.1. First, let us recall the explicit meaning of Langlands parameters, following [**22**] and [**30**].

Let σ be an $\mathbf{e}$-point of $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}$, i.e., a Weil $\check{G}$-local system on X. We obtain a map:

$$\mathrm{ev}_\sigma : \mathcal{E}xc_{\check{G}} \to \mathbf{e}$$

sending a function $f \in \mathcal{E}xc = \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \mathcal{O})$ to its value at σ. We abuse notation in also letting ev_σ denote the induced map (obtained by passing to H^0) $H^0(\mathcal{E}xc) \to \mathbf{e}$ of classical commutative algebras. We let $\mathfrak{m}_\sigma \subseteq H^0(\mathcal{E}xc)$ denote the corresponding maximal ideal.

Now recall (from [**4**], building on [**22**] and [**30**]) that $\mathcal{E}xc$ – hence $H^0(\mathcal{E}xc)$ – acts on $\mathcal{A}ut_{G,c}^{\mathrm{unr}}$.

We then define:

$$\mathcal{A}ut_{G,c,[\sigma]}^{\mathrm{unr}} \subseteq \mathcal{A}ut_{G,c}^{\mathrm{unr}}$$

to be the $\mathfrak{m}_\sigma$-torsion in the right hand side, i.e., $\psi \in \mathcal{A}ut_{G,c,[\sigma]}^{\mathrm{unr}}$ if $\mathfrak{m}_\sigma^n \cdot \psi = 0$ for $n \gg 0$.

We let $\mathcal{A}ut_{G,\mathrm{cusp},[\sigma]}^{\mathrm{unr}} := \mathcal{A}ut_{G,c,[\sigma]}^{\mathrm{unr}} \cap \mathcal{A}ut_{G,\mathrm{cusp}}^{\mathrm{unr}}$.

WARNING 3.6.1.1. Because $\mathcal{A}ut_{G,\mathrm{cusp}}^{\mathrm{unr}}$ is[33] finite-dimensional, $\mathcal{A}ut_{G,\mathrm{cusp}}^{\mathrm{unr}}$ decomposes as a direct sum:

$$\mathcal{A}ut_{G,\mathrm{cusp}}^{\mathrm{unr}} \simeq \bigoplus_{\sigma/\sim} \mathcal{A}ut_{G,\mathrm{cusp},[\sigma]}^{\mathrm{unr}}. \tag{3.13}$$

---

[31] Note that (3.8) predicts that $\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega_{\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}})$ lies in cohomological degree 0. Although we ultimately will be assuming restricted geometric Langlands, we are striving here to formulate a theorem independent of it, so we have included this statement.

[32] This assumption somewhat simplifies the discussion. Suitably formulated, the results here apply as well for general reductive groups.

[33] We remind that (using that G is semi-simple) there is a quasi-compact open $\mathcal{U} \subseteq \mathrm{Bun}_G$ defined over $\mathbf{F}_q$ such that any $\psi \in \mathcal{A}ut_{G,\mathrm{cusp}}^{\mathrm{unr}}$ vanishes outside $\mathcal{U}(\mathbf{F}_q)$ (see [**9**] Proposition 1.4.6 in the sheaf-theoretic setting); as $\mathcal{U}(\mathbf{F}_q)$ is finite, we clearly obtain the assertion.

(Here the implied equivalence relation $\sim$ relates σ_1 and σ_2 when $\mathfrak{m}_{\sigma_1} = \mathfrak{m}_{\sigma_2}$; according to [**22**] Proposition 0.38, this occurs exactly when σ_1 and σ_2 have equivalent semi-simplifications.)

However, we *do not* have a similar decomposition (3.13) for compactly supported automorphic functions; there are such functions that do not lie in any $\mathcal{A}ut^{\mathrm{unr}}_{G,c,[\sigma]}$.

3.6.2. Recall[34] that $\text{ps-Eis}(\mathcal{A}ut^{\mathrm{unr}}_{T,c}) \cap \mathcal{A}ut^{\mathrm{unr}}_{G,\mathrm{cusp}} = 0$.

Therefore, it suffices to show that any $\psi \in \mathcal{A}ut^{\mathrm{unr}}_{G,c,[\mathrm{triv}]} \subseteq \mathcal{A}ut^{\mathrm{unr}}_{G,c}$ lies in the image of the map $\text{ps-Eis} : \mathcal{A}ut^{\mathrm{unr}}_{T,c} \to \mathcal{A}ut^{\mathrm{unr}}_{G,c}$. This will be our objective.

3.6.3. Recall that we have $\delta_{\check{G}} \in \mathcal{E}xc_{\check{G}} := \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}}, \mathcal{O})$. As $\mathcal{A}ut^{\mathrm{unr}}_{\check{G},c}$ is acted on by $\mathcal{E}xc_{\check{G}}$, we may invert the action of $\delta_{\check{G}}$:

$$\mathcal{A}ut^{\mathrm{unr}}_{\check{G},c}[\delta_{\check{G}}^{-1}] := \underset{\delta_{\check{G}}\cdot -}{\mathrm{colim}}\, \mathcal{A}ut^{\mathrm{unr}}_{\check{G},c}.$$

We now translate from spectral Eisenstein series using restricted geometric Langlands (and Proposition 3.4.3.1). By Lemma 3.5.1.1 (1), the map ps-Eis intertwines the actions of $\delta_{\check{T}}$ and $\delta_{\check{G}}$ on $\mathcal{A}ut^{\mathrm{unr}}_{T,c}$ and $\mathcal{A}ut^{\mathrm{unr}}_{G,c}$ respectively. Moreover, the induced map:

$$\text{ps-Eis} : \mathcal{A}ut^{\mathrm{unr}}_{\check{T},c}[\delta_{\check{T}}^{-1}] \to \mathcal{A}ut^{\mathrm{unr}}_{\check{G},c}[\delta_{\check{G}}^{-1}]$$

is surjective by Theorem 3.5.2.1.

This means that for our given[35] automorphic funciton ψ, there is an integer $n \geqslant 0$ so that $\delta_{\check{G}}^n \cdot \psi = \text{ps-Eis}(\psi_0)$ for some $\psi_0 \in \mathcal{A}ut^{\mathrm{unr}}_{T,c}$.

Let $\lambda \in \mathbf{e}$ be the value of $\delta_{\check{G}}$ at the trivial local system $\mathrm{triv} \in \mathrm{LS}^{\mathrm{arthm}}_{\check{G}}$. Note that $(\delta_{\check{G}} - \lambda) \in \mathfrak{m}_{\mathrm{triv}} \subseteq H^0(\mathcal{E}xc)$, so for $m \gg 0$, we have:

$$(\delta_{\check{G}} - \lambda)^m \cdot \psi = 0.$$

By assumption, $\lambda \neq 0$. Therefore, we can find a polynomial $q(t) \in \mathbf{e}[t]$ with:

$$q(t) \cdot t^n = 1 \mod (t - \lambda)^m.$$

Then we clearly obtain:

$$\text{ps-Eis}(q(\delta_{\check{T}}) \cdot \psi_0) = q(\delta_{\check{G}}) \cdot \text{ps-Eis}(\psi_0) = q(\delta_{\check{G}}) \cdot \delta_{\check{G}}^n \cdot \psi = \psi.$$

This concludes the argument.

3.7. A toy model for Theorem 3.5.2.1. We now give a simpler setting in which a form of Theorem 3.5.2.1 holds. We will ultimately reduce the proof of Theorem 3.5.2.1 to this special case. The special case we consider is a standard result about the Grothendieck-Springer resolution.

3.7.1. *Analogies.* By way of analogy, we replace the diagram:

$$\begin{tikzcd} & \mathrm{LS}^{\mathrm{restr}}_{\check{B}} \arrow[dl, "\mathfrak{q}"'] \arrow[dr, "\mathfrak{p}"] & \\ \mathrm{LS}^{\mathrm{restr}}_{\check{T}} & & \mathrm{LS}^{\mathrm{restr}}_{\check{G}} \end{tikzcd}$$

with the diagram:

(3.14)
$$\begin{tikzcd} & \mathbf{B}\check{B} \arrow[dl] \arrow[dr] & \\ \mathbf{B}\check{T} & & \mathbf{B}\check{G}. \end{tikzcd}$$

[34] See [**26**] II.2.4 for a much stronger assertion.

[35] To be clear: this is true for *any* compactly supported automorphic function, but may be essentially vacuous if the form has another Langlands parameter.

In place of Frobenius, we consider each term in (3.14) with its identity endomorphism. Passing to fixed points under this map, we obtain the diagram:

$$\begin{tikzcd} & \check{B} \overset{\mathrm{ad}}{/} \check{B} \arrow[dl] \arrow[dr] & \\ \check{T} \overset{\mathrm{ad}}{/} \check{T} & & \check{G} \overset{\mathrm{ad}}{/} \check{G} \end{tikzcd}$$

Here for an algebraic group H, $H \overset{\mathrm{ad}}{/} H$ denotes the (stack) quotient of H acting on itself by conjugation; we remind[36] that $H \overset{\mathrm{ad}}{/} H \simeq (\mathbf{B}H)^{\mathbf{S}^1} = (\mathbf{B}H)^{\mathrm{id}=\mathrm{id}}$. We remark that the natural map $\mathcal{O}_{H \overset{\mathrm{ad}}{/} H} \to \omega_{H \overset{\mathrm{ad}}{/} H}$ is an isomorphism.

Therefore, the formalism of §3.2 yields a canonical map:

$$\mathrm{Eis}^{\mathrm{spec,toy}} : \Gamma(\check{T} \overset{\mathrm{ad}}{/} \check{T}, \mathcal{O}_{\check{T} \overset{\mathrm{ad}}{/} \check{T}}) \to \Gamma(\check{G} \overset{\mathrm{ad}}{/} \check{G}, \mathcal{O}_{\check{G} \overset{\mathrm{ad}}{/} \check{G}}). \tag{3.15}$$

LEMMA 3.7.1.1. *The left and right hand sides of* (3.15) *are concentrated in cohomological degree* 0 *and the map* $\mathrm{Eis}^{\mathrm{spec,toy}}$ *is surjective.*

PROOF. The most straightforward proof is as follows. We identify $\mathsf{Rep}(\check{T})$ with $\oplus_{\check{\lambda} \in \check{\Lambda}} \mathsf{Vect}$ and $\mathsf{Rep}(\check{G})$ with $\oplus_{\check{\lambda} \in \check{\Lambda}^+} \mathsf{Vect}$; here $\oplus$ denotes the coproduct on $\mathsf{DGCat}_{\mathrm{cont}}$ and we have implicitly chosen representatives of isomorphism classes of irreducible representations. Under this identification, we obtain canonical isomorphisms:

$$\begin{gathered} \oplus_{\check{\lambda} \in \check{\Lambda}} k \simeq \mathrm{tr}_{\mathsf{Rep}(\check{T})}(\mathrm{id}) (\simeq \Gamma(\check{T} \overset{\mathrm{ad}}{/} \check{T}, \mathcal{O}_{\check{T} \overset{\mathrm{ad}}{/} \check{T}})) \\ \oplus_{\check{\lambda} \in \check{\Lambda}^+} k \simeq \mathrm{tr}_{\mathsf{Rep}(\check{G})}(\mathrm{id}) (\simeq \Gamma(\check{G} \overset{\mathrm{ad}}{/} \check{G}, \mathcal{O}_{\check{G} \overset{\mathrm{ad}}{/} \check{G}})). \end{gathered} \tag{3.16}$$

These identities clearly imply that both sides of (3.15) are concentrated in degree 0. We let $e_{\check{T},\check{\lambda}}$ ($\check{\lambda} \in \check{\Lambda}$) and $e_{\check{G},\check{\lambda}}$ ($\check{\lambda} \in \check{\Lambda}^+$) denote the basis vectors for these vector spaces coming from the displayed isomorphism. By Borel-Weil-Bott, for $\check{\lambda}$ dominant, the map $\mathsf{Rep}(\check{T}) \to \mathsf{Rep}(\check{G})$ sends $\ell^{w_0(\check{\lambda})} \in \mathsf{Rep}(\check{T})^\heartsuit$ (the 1-dimensional representation corresponding to $w_0(\check{\lambda})$ to $V^\lambda) \in \mathsf{Rep}(\check{G})$ (the representation with highest weight λ), so sends $e_{\check{T},w_0(\check{\lambda})}$ to $e_{\check{G},\lambda}$; this yields the surjectivity.

□

REMARK 3.7.1.2. We remark (although we do not need it) that in (3.16), the isomorphism $\oplus_{\check{\lambda} \in \check{\Lambda}^+} k \simeq \Gamma(\check{G} \overset{\mathrm{ad}}{/} \check{G}, \mathcal{O}_{\check{G} \overset{\mathrm{ad}}{/} \check{G}})$ sends $e_{\check{G},\check{\lambda}}$ (notation as before) to the trace function corresponding to the representation $V^{\check{\lambda}}$ of $\check{G}$, and similarly for $\check{T}$. Therefore, the composition:

$$\bigoplus_{\check{\lambda} \in \check{\Lambda}} k \simeq \Gamma(\check{T} \overset{\mathrm{ad}}{/} \check{T}, \mathcal{O}_{\check{T} \overset{\mathrm{ad}}{/} \check{T}}) \xrightarrow{\mathrm{Eis}^{\mathrm{spec,toy}}} \Gamma(\check{G} \overset{\mathrm{ad}}{/} \check{G}, \mathcal{O}_{\check{G} \overset{\mathrm{ad}}{/} \check{G}}) \simeq \mathrm{Fun}(\check{T})^W = \Big(\bigoplus_{\check{\lambda} \in \check{\Lambda}} k \Big)^W$$

is explicitly calculated using the Weyl character formula (and Borel-Weil-Bott).

[36] There is some sign ambiguity in the isomorphism here. Usually this does not matter, as for the present discussion. But it will matter later in the paper. We clarify the implied sign conventions in §4.4.2.

3.7.2. We conclude by recording a variant.

Note that we have a commutative diagram:

$$\begin{tikzcd} & \check{B}\overset{\mathrm{ad}}{/}\check{B} \arrow[dl] \arrow[dr] & \\ \check{T}\overset{\mathrm{ad}}{/}\check{T} \arrow[dr] & & \check{G}\overset{\mathrm{ad}}{/}\check{G} \arrow[dl] \\ & \check{T}/\!/W & \end{tikzcd}$$

where $\check{T}/\!/W = \mathrm{Spec}(\mathbf{e}[\check{\Lambda}]^W)$ is the GIT quotient, the lower left map is tautological and the lower right map is the standard characteristic polynomial map (uniquely characterized by this diagram).

Therefore, $\mathrm{Eis}^{\mathrm{spec,toy}}$ is a map of $\mathrm{Fun}(\check{T}/\!/W)$-modules. Lemma 3.7.1.1 then says that $\mathrm{Eis}^{\mathrm{spec,toy}}$ is an epimorphism of $\mathrm{Fun}(\check{T}/\!/W)$-modules, so we obtain:

Corollary 3.7.2.1. *For any $g \in \Gamma(\check{T}/\!/W, \mathcal{O}_{\check{T}/\!/W})$, the map:*

$$\mathrm{Eis}^{\mathrm{spec,toy}} : \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T}, \mathcal{O}_{\check{T}\overset{\mathrm{ad}}{/}\check{T}})[g^{-1}] \to \Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G}, \mathcal{O}_{\check{G}\overset{\mathrm{ad}}{/}\check{G}})[g^{-1}].$$

is a surjection.

4. Grothendieck-Springer theory for $\mathrm{LS}^{\mathrm{arthm}}$

The goal of this section is to prove Theorem 3.5.2.1. As this theorem occurs purely on the spectral side, throughout this section, we work by default over the field $\mathbf{e}$.

4.1. Base-points and Weil group notation. Below, we take $x_0 \in X(k)$ a marked geometric point, which will serve as the base-point of our fundamental group; here we remind that $k = \overline{\mathbf{F}}_q$.

We encourage the reader to be kind to themselves and assume that x_0 is defined over $\mathbf{F}_q$; in this case essentially all of the remaining material of §4.1 can be ignored.

With that said, we include some technical material here to allow for the case where X_0 has no rational points.

4.1.1. Let $\widetilde{X}$ denote the universal cover of X based at x_0; by definition, $\widetilde{X}$ is connected, pro-finite étale over X, equipped with a lift $\widetilde{x}_0$ of x_0, and initial among all such data. Note that $\widetilde{X}$ is also the universal cover of X_0, so there is a tautological action of $\pi_1^{\text{ét}}(X_0)$ ($= \pi_1^{\text{ét}}(X_0, x_0)$) on $\widetilde{X}$ (realizing it as a $\pi_1^{\text{ét}}(X)$-torsor over X and a $\pi_1^{\text{ét}}(X_0)$-torsor over X_0, each torsor being understood as locally trivial for the pro-étale topology).

Let $\mathrm{Frob}_X : X \to X$ be the geometric Frobenius map. Choose once and for all a lift of the point $\mathrm{Frob}_X(x_0)$ to $\widetilde{X}$. It is easy to see that there is a unique map $\widetilde{\mathrm{Frob}}_X$ fitting into the commutative diagram

$$\begin{tikzcd} \widetilde{X} \arrow[r, dotted, "\widetilde{\mathrm{Frob}}_X"] \arrow[d, "\pi"] & \widetilde{X} \arrow[d, "\pi"] \\ X \arrow[r, "\mathrm{Frob}_X"] & X \end{tikzcd}$$

and sending $\widetilde{x}_0$ to our chosen lift of $\mathrm{Frob}_X(x_0)$ (which will now be denoted $\widetilde{\mathrm{Frob}}_X(\widetilde{x}_0)$).

This choice also (relatedly) defines an action of $\widehat{\mathbf{Z}}$ on $\widetilde{X}$ in $\mathsf{Sch}_{/X_0}$; the inverse[37] to generator $-1 \in \widehat{\mathbf{Z}}$ acts by a map $\gamma : \widetilde{X} \to \widetilde{X}$ characterized by being Frobenius semi-linear over k and so that $\gamma \circ \widetilde{\mathrm{Frob}}_X$ is the absolute Frobenius of $\widetilde{X}$.

This data defines a splitting of the map $\mathcal{W}_X \to \mathbf{Z}$. We let $F \in \mathcal{W}_X$ denote the image of $1 \in \mathbf{Z}$ under the splitting. By definition, $F^{-1} \in \mathcal{W}_X \subseteq \pi_1^{\text{ét}}(X_0)$ acts on $\widetilde{X}$ by the map denoted γ above.

[37]We note that per our conventions, the inverse to the generator $-1 \in \widehat{\mathbf{Z}}$ corresponds to the arithmetic Frobenius when we identify $\widehat{\mathbf{Z}} \simeq \mathrm{Gal}(\mathbf{F}_q)$.

For $g \in \pi_1^{\text{ét}}(X)$, we let ${}^F g := \mathrm{Ad}_F(g)$. Note that the choice of point $\widetilde{\mathrm{Frob}}_X(\widetilde{x}_0)$ gives an isomorphism $\pi_1^{\text{ét}}(X, \mathrm{Frob}_X(x_0)) \simeq \pi_1^{\text{ét}}(X, x_0)$, and the composition:

$$\pi_1^{\text{ét}}(X, x_0) \xrightarrow{\pi_1^{\text{ét}}(\mathrm{Frob}_X)} \pi_1^{\text{ét}}(X, \mathrm{Frob}_X(x_0)) \simeq \pi_1^{\text{ét}}(X, x_0)$$

is the map $g \mapsto {}^F g$,[38] i.e., we have:

$$\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g) = {}^F g. \tag{4.1}$$

4.1.2. The choice of point $\widetilde{\mathrm{Frob}}_X(\widetilde{x}_0)$ also defines an isomorphism:

$$x_0^* \simeq \mathrm{Frob}_X(x_0)^* : \mathsf{Lisse}(X)^\heartsuit \to \mathsf{Vect}^\heartsuit$$

of $\mathbf{e}$-linear symmetric monoidal functors. In fact, we claim that this comes from an isomorphism of symmetric monoidal DG functors:

$$x_0^* \simeq \mathrm{Frob}_X(x_0)^* : \mathsf{qLisse}(X) \to \mathsf{Vect}. \tag{4.2}$$

Indeed, this follows formally whenever $\mathsf{qLisse}(X)$ is the derived category of its heart. This is the case for $X \neq \mathbf{P}^1$ by [**2**] Theorem E.2.8. Slightly more elementarily (and allowing genus 0), we choose $U \subseteq X$ affine open and containing x_0 and $\mathrm{Frob}_X(x_0)$, and then $\mathsf{qLisse}(U)$ is the derived category of its heart by the (simpler) Theorem E.2.8 (a).

Remark 4.1.2.1. The following remark will not be used. For the present moment, let k be *any* algebraically closed field (not just $\overline{\mathbf{F}}_q$). Let Y/k be a connected scheme of finite type. Let $y_1, y_2 \in Y(k)$ be two points. Then at this moment, it is natural to ask if there exists an isomorphism of symmetric monoidal DG functors:

$$y_1^* \simeq y_2^* : \mathsf{qLisse}(Y) \to \mathsf{Vect}.$$

We claim this is so. Indeed, we have effectively treated above the case of a smooth connected curve. The case of any connected curve follows in an evident way by considering normalizations (using intersection points between irreducible components of the singular curve as signposts leading the way). Finally, the general case follows by noting that there exists a connected curve C and a map $C \to Y$ with y_1 and y_2 in its image by an elementary argument. (One wonders if there is a purely Tannakian argument that would apply in this derived setup.)

4.2. The adjoint quotient. Let H be an affine algebraic group in what follows.

4.2.1. Let $\mathrm{LS}_H^{\mathrm{restr},\circ}$ denote the *neutral* connected component of $\mathrm{LS}_H^{\mathrm{restr}}$, i.e., the connected component containing the trivial H-local system on X. We remind from [**2**] Proposition 3.7.2 that $\mathrm{LS}_H^{\mathrm{restr}}$ parametrizes (in a precise sense) H-local systems on X with trivial semi-simplification; in what follows, we refer to these as *unipotent* H-local systems.

We then set:

$$\mathrm{LS}_H^{\mathrm{arthm},\circ} := \mathrm{LS}_H^{\mathrm{arthm}} \underset{\mathrm{LS}_H^{\mathrm{restr}}}{\times} \mathrm{LS}_H^{\mathrm{restr},\circ}.$$

In other words, $\mathrm{LS}_H^{\mathrm{arthm},\circ}$ is the fixed points of Frobenius acting on $\mathrm{LS}_H^{\mathrm{restr},\circ}$; it may be thought of as parametrizing Weil H-local systems that are *geometrically* unipotent.

[38]Indeed, we have $\widetilde{\mathrm{Frob}}_X(g\widetilde{x}_0) = \pi_1^{\text{ét}}(\mathrm{Frob}_X)(g) \cdot \widetilde{\mathrm{Frob}}_X(\widetilde{x}_0)$ by definition of $\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g)$.

Now by definition of $F = \gamma^{-1}$, we have $F^{-1} \cdot \widetilde{\mathrm{Frob}}_X(g\widetilde{x}_0) = \Phi_{\widetilde{X}}(g\widetilde{x})$ for $\Phi_{\widetilde{X}}$ the absolute Frobenius. By functoriality, absolute Frobenius is a map of spaces with $\pi_1^{\text{ét}}(X)$-actions, so $\Phi_{\widetilde{X}}(g\widetilde{x}_0) = g \cdot \Phi_{\widetilde{X}}(\widetilde{x}_0)$.

Comparing to our earlier equation, we see this expression equals $F^{-1}\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g) \cdot \widetilde{\mathrm{Frob}}_X(\widetilde{x}_0) = F^{-1}\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g)F \cdot F^{-1}\widetilde{\mathrm{Frob}}_X(\widetilde{x}_0) = F^{-1}\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g)F \cdot \Phi_{\widetilde{X}}(\widetilde{x}_0)$. Therefore, $g = \pi_1^{\text{ét}}(\mathrm{Frob}_X)(g)F$, yielding the assertion.

4.2.2. There is a canonical map $\mathbf{B}H \to \mathrm{LS}_H^{\mathrm{restr},\circ}$ corresponding to the trivial H-local system. This map is Frobenius equivariant, where Frobenius acts trivially on $\mathbf{B}H$.

Passing to Frobenius fixed points, we obtain a map:

$$\tau_H : H \overset{\mathrm{ad}}{/} H \to \mathrm{LS}_H^{\mathrm{arthm},\circ}. \tag{4.3}$$

Remark 4.2.2.1. Informally, the composition $H \to H \overset{\mathrm{ad}}{/} H \to \mathrm{LS}_H^{\mathrm{arthm},\circ}$ sends $h \in H$ to the Weil representation $\mathcal{W}_X \to H$ defined by $\mathcal{W}_X \twoheadrightarrow \mathcal{W}_X/\pi_1^{\text{ét}}(X) \overset{\mathrm{Fr}}{\simeq} \mathbf{Z} \xrightarrow{1\mapsto h} H$. As we only consider the *groupoid* of Weil representations, this map factors through the adjoint quotient as desired.

Remark 4.2.2.2. One main idea below is that τ_H is not too far from being an isomorphism. To motivate what follows, we observe the following obstruction to τ_H being an isomorphism. Coarsely (e.g., at the level of field-valued points), the enemy is clearly Weil representations that are geometrically unipotent and geometrically non-trivial.

Suppose $H = GL_2$. Let $\lambda \in \mathbf{e}^\times$ and let σ_λ denote the 1-dimensional Weil group representation where Frobenius acts as multiplication by λ (and $\pi_1^{\text{ét}}(X)$ acts trivially). Extensions $0 \to \sigma_\lambda \to \sigma \to \sigma_1 \to 0$ are classified by suitable group cohomology for $\mathbf{Z}$, i.e., by H^1 of the complex:

$$\underline{\mathrm{Hom}}_{\mathbf{e}[\mathbf{Z}]\text{-}\mathsf{mod}}(\mathrm{C}_{\text{ét},\bullet}(X), \sigma_\lambda).$$

Here $\mathrm{C}_{\text{ét},\bullet}(X)$ is the complex of *étale homology* for X, and the $\mathbf{Z}$-action has generator acting by geometric Frobenius on étale homology. If $\lambda \neq 1$, it is easy to see that we have an exact sequence:

$$0 \to H^1(\underline{\mathrm{Hom}}_{\mathbf{e}[\mathbf{Z}]\text{-}\mathsf{mod}}(\mathrm{C}_{\text{ét},\bullet}(X), \sigma_\lambda)) \to H^1_{\text{ét}}(X, \sigma_\lambda) \xrightarrow{\phi_X^{-1}-\mathrm{id}} H^1_{\text{ét}}(X, \sigma_\lambda)$$

where ϕ_X is the geometric Frobenius acting on $H^1_{\text{ét}}$.[39] Here we have $H^1_{\text{ét}}(X, \sigma_\lambda) = H^1_{\text{ét}}(X, \mathbf{e})$, but with Frobenius action given as λ times the standard one. Therefore, if λ is a Frobenius eigenvalue appearing in $H^1_{\text{ét}}(X, \mathbf{e})$, we find *geometrically* non-trivial extensions of the desired type.

4.2.3. *Splitting.* We obtain a map $\mathrm{LS}_H^{\mathrm{restr}} \to \mathbf{B}H$ by restriction to x_0. This map intertwines Frobenius with the identity by[40] §4.1.2, so on fixed points we obtain a map:

$$\mathrm{LS}_H^{\mathrm{arthm}} \to H \overset{\mathrm{ad}}{/} H.$$

We denote this map by $\chi_H = \chi_{H,x_0}$, and similarly its restriction to $\mathrm{LS}_H^{\mathrm{arthm},\circ}$.

By construction, the composition:

$$H \overset{\mathrm{ad}}{/} H \xrightarrow{\tau_H} \mathrm{LS}_H^{\mathrm{arthm}} \xrightarrow{\chi_H} H \overset{\mathrm{ad}}{/} H$$

is the identity map.

4.3. Non-resonance.

4.3.1. Define the set $\mathcal{R}_X \subseteq \mathbf{e}^\times$ as the set of of eigenvalues of the (geometric) Frobenius acting on $H^1_{\text{ét}}(X, \mathbf{e}) \times H^2_{\text{ét}}(X, \mathbf{e})$.

Remark 4.3.1.1. By the Weil conjectures for curves, $1 \notin \mathcal{R}_X$. Also, q always lies in $\mathcal{R}_X$ (but this is less relevant to us at the present moment).

[39] Note that geometric Frobenius for homology and cohomology are transpose (i.e., dual) morphisms. However, if we consider, say, homology $H_{\text{ét},1}(X) = (\pi_1^{\text{ét}}(X)^{\mathrm{ab}})^\wedge_\ell \otimes \mathbf{e}$ with its geometric Frobenius (which corresponds to $\mathrm{Ad}_F : \pi_1^{\text{ét}}(X) \to \pi_1^{\text{ét}}(X)$ by (4.1)) as a $\mathbf{Z}$-representation, the *dual* $\mathbf{Z}$-action on cohomology has the generator acting by *arithmetic* Frobenius. This accounts for the inverse sign in the above formula.

[40] To be clear, when x_0 was not $\mathbf{F}_q$-rational, this equivariance depended on auxiliary choices.

4.3.2. Let H be an affine algebraic group.

Let V be a finite-dimensional representation of H. Let $\rho_V : H \to GL(V)$ be the corresponding homomorphism. Let $\mathrm{ch}_V : H\overset{\mathrm{ad}}{/}H \times \mathbf{A}^1 \to \mathbf{A}^1$ be the map fitting into a commutative diagram:

$$\begin{tikzcd} H \times \mathbf{A}^1 \arrow[r, "\rho_V \times \mathrm{id}"] \arrow[d] & GL(V) \times \mathbf{A}^1 \arrow[d, "(g,\lambda)\mapsto \det(g-\lambda\cdot \mathrm{id}_V)"] \\ H\overset{\mathrm{ad}}{/}H \times \mathbf{A}^1 \arrow[r, "\mathrm{ch}_V"] & \mathbf{A}^1. \end{tikzcd}$$

Explicitly, for $h \in H$ and $\lambda \in \mathbf{A}^1$, $\mathrm{ch}_V([h], \lambda)$ is the characteristic polynomial of $\rho_V(h)$ evaluated at λ.

We then let $(H\overset{\mathrm{ad}}{/}H)^{\text{non-res}} \subseteq H\overset{\mathrm{ad}}{/}H$ denote the open consisting of conjugacy classes $[h]$ such that $\prod_{\lambda \in \mathcal{R}_X} \mathrm{ch}_{\mathfrak{h}}([h], \lambda) \neq 0$, where $\mathfrak{h}$ is the adjoint representation of H. Explicitly, $(H\overset{\mathrm{ad}}{/}H)^{\text{non-res}}$ is the set of conjugacy classes $[h]$ such that the matrix $\rho_{\mathfrak{h}}(h) \in GL(\mathfrak{h})$ does not have any eigenvalues in $\mathcal{R}_X$. We remark that the open embedding $(H\overset{\mathrm{ad}}{/}H)^{\text{non-res}} \hookrightarrow H\overset{\mathrm{ad}}{/}H$ is clearly affine.

Remark 4.3.2.1. Note that $[1] \in (H\overset{\mathrm{ad}}{/}H)^{\text{non-res}}$ by Remark 4.3.1.1.

Notation 4.3.2.2. For any stack $\mathcal{Y}$ equipped with a structure map to $(H\overset{\mathrm{ad}}{/}H)^{\text{non-res}}$, we let $\mathcal{Y}^{\text{non-res}} := \mathcal{Y} \times_{H\overset{\mathrm{ad}}{/}H} (H\overset{\mathrm{ad}}{/}H)^{\text{non-res}}$. We use this notation particularly in the case $\mathcal{Y} = \mathrm{LS}_H^{\mathrm{arthm},\circ}$ equipped with the structural map χ_H.

4.3.3. *Main geometric result.* The following result compares arithmetic local systems with the adjoint quotient:

Theorem 4.3.3.1. *The map:*

$$\tau_H : (H\overset{\mathrm{ad}}{/}H)^{\text{non-res}} \to \mathrm{LS}_H^{\mathrm{arthm},\circ,\text{non-res}}$$

from (4.3) *is an isomorphism.*

The proof of this result is the subject of §4.4.

4.4. Proof of Theorem 4.3.3.1.

4.4.1. *A criterion for a map to be an isomorphism.* We begin by observing:

Lemma 4.4.1.1. *Let $f : \mathcal{Y}_1 \to \mathcal{Y}_2$ be a morphism of algebraic stacks that are locally almost of finite type (over the algebraically closed field* $\mathbf{e}$*).*

Then f is an isomorphism if and only if:

(i) f is formally étale, i.e., its cotangent complex $\Omega^1_{\mathcal{Y}_1/\mathcal{Y}_2} \in \mathsf{QCoh}(\mathcal{Y}_1)$ vanishes.

(ii) The map $\mathcal{Y}_1(\mathbf{e}) \to \mathcal{Y}_2(\mathbf{e})$ is an isomorphism of (1-)groupoids.

Here we explicitly remark that condition (ii) *can be separated into the two separate conditions:*

(ii_1) For every $y_1 \in \mathcal{Y}_1(\mathbf{e})$, the map $\mathrm{Aut}_{\mathcal{Y}_1(\mathbf{e})}(y_1) \to \mathrm{Aut}_{\mathcal{Y}_2(\mathbf{e})}(f(y_1))$ of automorphism groups is an isomorphism.[41]

(ii_2) For every $y_2 \in \mathcal{Y}_2(\mathbf{e})$, there exists $y_1 \in \mathcal{Y}_1(\mathbf{e})$ and an isomorphism $f(y_1) \simeq y_2 \in \mathcal{Y}_2(\mathbf{e})$.

Proof. It suffices to show that for every affine S locally almost of finite type and equipped with a map $S \to \mathcal{Y}_2$, the map $S \times_{\mathcal{Y}_2} \mathcal{Y}_1 \to S$ is an isomorphism. The properties (i) and (ii) are obviously preserved under such base-change, so we may assume $\mathcal{Y}_2$ is an affine scheme. Moreover, it is standard that $\mathcal{Y}_1 \to \mathcal{Y}_2$ is an isomorphism if and only if $\mathcal{Y}_1 \times_{\mathcal{Y}_2} \mathcal{Y}_2^{\mathrm{cl}} \to \mathcal{Y}_2^{\mathrm{cl}}$ is so; therefore, we may assume $\mathcal{Y}_2$ is moreover classical.

Now $\mathcal{Y}_1$ is an algebraic stack with trivial automorphism groups at $\mathbf{e}$-points, and therefore an algebraic space. Moreover, $\mathcal{Y}_1 \to \mathcal{Y}_2$ is étale, so $\mathcal{Y}_1$ is also classical. Now f is a radicial map (because it is locally of

[41] We emphasize that there is no room for anything derived here; this is a map between two sets.

finite type and injective on $\mathbf{e}$-points) and étale, so an open embedding. Finally, because f is surjective on $\mathbf{e}$-points, it must be an isomorphism.

$\square$

Below, we will verify the above hypotheses for the map τ_H considered in Theorem 4.3.3.1.

4.4.2. *Conventions, formulae, and signs.* Before proceeding, we establish certain signs that will be important. Roughly speaking, it is conceptually difficult to distinguish $\tau_H([h])$ from $\tau_H([h^{-1}])$, but Theorem 4.3.3.1 does distinguish them, so we must explain exactly how to understand the map τ_H a bit more explicitly. (The reader is invited to skip this digression and return to it as needed.)

Below, we let $\mathrm{Frob}_X : X \to X$ denote the geometric Frobenius map.

We normalize lisse Weil sheaves on X to be a pair (σ, α) where $\sigma \in \mathsf{Lisse}(X)^{\heartsuit} = \mathsf{qLisse}(X)^{\heartsuit}$ is equipped with an isomorphism $\alpha : \sigma \xrightarrow{\simeq} \mathrm{Frob}_X^*(\sigma)$. The direction of the map α is the "sign" in question. Let us explain first why (and in what sense) this sign is the right one for our existing conventions.

Note that $V := x_0^*(\sigma)$ is a representation ρ^{geom} of $\pi_1^{\text{ét}}(X)$. We also obtain an isomorphism:

$$V = x_0^*(\sigma) \xrightarrow{\alpha_{x_0}} \mathrm{Frob}_X(x_0)^*(\sigma) \overset{(4.2)}{\simeq} x_0^*(\sigma) = V \tag{4.4}$$

that we denote by $\rho(F)$. In the notation of §4.1.1, one finds tautologically that $\rho(F) \circ \rho^{\mathrm{geom}}(g) = \rho^{\mathrm{geom}}(\pi_1^{\text{ét}}(\mathrm{Frob}_X)(g)) \circ \rho(F)$. By (4.1), we can rewrite this equation as:

$$\rho(F)\rho^{\mathrm{geom}}(g)\rho(F)^{-1} = \rho^{\mathrm{geom}}(FgF^{-1})$$

so we obtain representation of $\mathcal{W}_X$ on V with F acting by (4.4) – had α gone the other way, we would need to invert (4.4).

Similarly, for H an affine algebraic group, a Weil H-local system is an H-local system σ_H on X with an isomorphism $\alpha : \sigma_H \xrightarrow{\simeq} \mathrm{Frob}_X^*(\sigma_H)$ (of H-local systems). As a consequence, for $h \in H$, $\tau_H([h])$ has σ_H trivial and α is given as multiplication by h. This ensures that the corresponding Weil group representation $\mathcal{W}_X \to H(\mathbf{e})$ factors through $\mathbf{Z} = \mathcal{W}_X/\pi_1^{\text{ét}}(X)$ and sends the generator to h (as it was supposed to).

Finally, for a lisse Weil sheaf (σ, α), the natural "geometric" Frobenius action ϕ_σ on its cohomology is given by the operator:

$$\mathrm{C}_{\text{ét}}(X, \sigma) \to \mathrm{C}_{\text{ét}}(X, \mathrm{Frob}_X^*(\sigma)) \xrightarrow{\alpha^{-1}} \mathrm{C}_{\text{ét}}(X, \sigma) \tag{4.5}$$

where the first map is the tautological one.

Remark 4.4.2.1. We wish to be clear about the logical status of the above material. First, we have argued that the map α should be considered as going in a certain direction. But at some level, this is a moral argument, not a mathematical one. Rather, we have *made explicit* a certain[42] convention that was implicit before (and shown how it leads to the orientation informally suggested in Remark 4.2.2.1). Logically speaking, establishing this convention was strictly necessary for the statement of Theorem 4.3.3.1.

4.4.3. τ_H *is formally étale.* We will show that $\tau_H|_{(H \overset{\mathrm{ad}}{/} H)^{\text{non-res}}}$ is formally étale.

First, note that we are reduced to checking that the tangent complex vanishes (e.g., both sides have perfect cotangent complexes). Moreover, we can check this on fibers at all $\mathbf{e}$-points as both sides are locally almost of finite type.

In general, for $\sigma \in \mathrm{LS}_H^{\mathrm{arthm}}(\mathbf{e})$, we can compute the tangent complex as:[43]

$$T_{\mathrm{LS}_H^{\mathrm{arthm}},\sigma} = \mathrm{C}_{\text{ét}}(X, \mathfrak{h}_\sigma[1])^{\mathbf{Z}} = \mathrm{Ker}\left(\mathrm{id} - \phi_\sigma : \mathrm{C}_{\text{ét}}(X, \mathfrak{h}_\sigma[1]) \to \mathrm{C}_{\text{ét}}(X, \mathfrak{h}_\sigma[1])\right).$$

Here $\mathfrak{h}_\sigma$ is the adjoint Weil local system on X induced by σ, $\mathrm{C}_{\text{ét}}(X, \mathfrak{h}_\sigma[1])$ is its étale cohomology complex (up to shift), and we are taking $\mathbf{Z}$-invariants with respect to the action of the Frobenius ϕ_σ (coming from the Weil structure on σ).

[42]For even more clarity: $H \overset{\mathrm{ad}}{/} H = (\mathbf{B}H)^{S^1}$ has an automorphism of "loop reversal," and we need to remove the ambiguity this automorphism provides.

[43]See [**2**] Proposition 2.2.2, §24.5.1.

For $h \in H(\mathbf{e})$, the tangent complex $T_{H\overset{\mathrm{ad}}{/}H,[h]} \in \mathsf{Vect}$ is the homotopy kernel (i.e., shifted cone):

$$T_{H\overset{\mathrm{ad}}{/}H,[h]} = \mathrm{Ker}(\mathrm{id} - \mathrm{Ad}_h : \mathfrak{h}[1] \to \mathfrak{h}[1])$$

where Ad_h is the adjoint action of h on $\mathfrak{h}$; more naturally, writing $H\overset{\mathrm{ad}}{/}H = \mathcal{M}aps(\mathbf{BZ}, \mathbf{B}H)$ and this formula yields $\mathbf{Z}$-invariants for the $\mathbf{Z}$-action on $\mathfrak{h}[1] = T_{\mathbf{B}H,\mathrm{Spec}(k)}$ with generator acting by Ad_h. (One consequence: we see that in the above description of $T_{H\overset{\mathrm{ad}}{/}H,[h]}$ could have used $\mathrm{Ad}_{h^{-1}}$ in place of Ad_h.)

On the other hand, we have:

$$\mathrm{C}_{\text{ét}}(X, \mathfrak{h}_{\tau_H([h])}[1]) = \mathrm{C}_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}[1]$$

with Frobenius:

$$\phi_{\tau_H([h])} = \phi_X \otimes \mathrm{Ad}_{h^{-1}}$$

where the inverse occurs because of the appearance of α^{-1} in (4.5).

The Frobenius equivariant map $\mathbf{e} = H^0(X, \mathbf{e}) \to \mathrm{C}_{\text{ét}}(X, \mathbf{e})$ (with Frobenius acting trivially on the source) induces a commutative diagram:

$$\begin{tikzcd}
\mathfrak{h}[1] = \mathbf{e} \otimes \mathfrak{h}[1] \arrow[r, "\mathrm{id} - \mathrm{Ad}_{h^{-1}}"] \arrow[d] & \mathfrak{h}[1] = \mathbf{e} \otimes \mathfrak{h}[1] \arrow[d] \\
\mathrm{C}_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}[1] \arrow[r, "\mathrm{id} - \phi_X \otimes \mathrm{Ad}_{h^{-1}}"] & \mathrm{C}_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}[1].
\end{tikzcd}$$

Passing to (homotopy) kernels along the rows yields the differential for τ_H at $[h]$.[44]

Therefore, we see that τ_H is formally étale at $[h]$ if and only if:

$$\mathrm{id} - \phi_X \otimes \mathrm{Ad}_{h^{-1}} : \tau^{\geqslant 1}\, \mathrm{C}_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h} \to \tau^{\geqslant 1}\, \mathrm{C}_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}$$

is an isomorphism, or equivalently, the induced maps on cohomology:

$$\mathrm{id} - \phi_X \otimes \mathrm{Ad}_{h^{-1}} : H^1_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h} \to H^1_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}$$
$$\mathrm{id} - \phi_X \otimes \mathrm{Ad}_{h^{-1}} : H^2_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h} \to H^2_{\text{ét}}(X, \mathbf{e}) \otimes \mathfrak{h}$$

are isomorphisms. Clearly this happens exactly when 1 is not an eigenvalue of $\phi_X \otimes \mathrm{Ad}_{h^{-1}}$, which occurs exactly when 1 cannot be written as $\lambda \cdot \mu$ for λ an eigenvalue of ϕ_X and μ an eigenvalue of $\mathrm{Ad}_{h^{-1}} = \mathrm{Ad}_h^{-1}$, i.e., when no eigenvalues of Ad_h lie in $\mathcal{R}_X$. This is the defining condition for $[h]$ to lie in $(H\overset{\mathrm{ad}}{/}H)^{\text{non-res}}$, so we obtain the claim.

4.4.4. *Stabilizers.* Next, we verify condition (ii_1) from Lemma 4.4.1.1. In fact, this is obvious, and we will never use the subtleties of non-resonance in this step. We explicitly spell out the argument here:

Suppose $\sigma \in \mathrm{LS}_H^{\mathrm{arthm}}(\mathbf{e})$. By definition, σ lifts to a continuous Weil group representation $\rho : \mathcal{W}_X \to H(\mathbf{e})$ that is well-defined up to conjugacy. In this case, $\mathrm{Aut}_{\mathrm{LS}_H^{\mathrm{arthm}}(\mathbf{e})}(\sigma)$ is the stabilizer of the image of ρ in $H(\mathbf{e})$.

Similarly, an $\mathbf{e}$-point in $H\overset{\mathrm{ad}}{/}H$ lifts to some $h \in H(\mathbf{e})$, and $\mathrm{Aut}_{H\overset{\mathrm{ad}}{/}H(\mathbf{e})}([h])$ is the stabilizer of h.

Now for $h \in H$, $\tau_H([h])$ is the Weil group representation $\mathcal{W}_X \to \mathbf{Z} \xrightarrow{1 \mapsto h} H(\mathbf{e})$, whose stabilizer obviously coincides with that of h.

4.4.5. *Lifting isomorphism classes: setup.* Finally, we verify (ii_2). Suppose $\sigma \in \mathrm{LS}_H^{\mathrm{arthm},\circ,\text{non-res}}(\mathbf{e})$. We lift σ to a continuous representation $\rho : \mathcal{W}_X \to H(\mathbf{e})$. Let $\rho_0 : \pi_1^{\text{ét}}(X) \to H(\mathbf{e})$ denote the restriction of ρ to the geometric fundamental group; our task is to show that ρ_0 is trivial.

This is a concrete linear algebra problem; we spell out the details below. We use the notation of §4.1.1 (in particular, $F \in \mathcal{W}_X$ and $g \mapsto {}^F g$). In addition, we introduce more notation:

- Let $H_\circ \subseteq H$ denote the Zariski closure of $\mathrm{Image}(\rho_0)$.
- Let $\theta : H_\circ \xrightarrow{\simeq} H_\circ$ denote the adjoint action of $\rho(F)$, i.e., $\theta(h) = \mathrm{Ad}_{\rho(F)}(h)$.

[44] To see this, consider $H\overset{\mathrm{ad}}{/}H$ as the moduli of arithmetic local systems on $\mathrm{Spec}(k)$, then apply the above discussion about $\mathrm{LS}_H^{\mathrm{arthm}}$ accordingly.

In these terms, note that we have:

$$\rho_0({}^F g) = \theta(\rho_0(g)) \text{ for all } g \in \pi_1^{\text{ét}}(X). \tag{4.6}$$

We now make the following additional observations about our hypotheses.

First, that ρ_0 defines a point in $\mathrm{LS}_H^{\mathrm{restr},\circ} \subseteq \mathrm{LS}_H^{\mathrm{restr}}$ means that ρ_0 factors through a unipotent subgroup of H by [**2**] Proposition 3.7.2. Equivalently, $H_\circ$ is unipotent.

Second, note that the non-resonance condition means that $\mathrm{Ad}_{\rho(F)} = \mathrm{Lie}(\theta) : \mathfrak{h} \to \mathfrak{h}$ has no eigenvalues in $\mathcal{R}_X$.

4.4.6. *Lifting isomorphism classes: proof.* In the above notation, our task is to show that $H_\circ$ is trivial. By unipotence, it suffices to show that its abelianization $H_\circ^{\mathrm{ab}}$ is trivial. Let $V := \mathrm{Lie}(H_\circ^{\mathrm{ab}})$; as $H_\circ^{\mathrm{ab}}$ is an abelian unipotent group, we abuse notation in identifying it with (the $\mathbf{e}$-scheme associated with) its Lie algebra.

By functoriality, θ induces an automorphism of V, which we also denote by θ. Suppose $V \neq 0$; then there exists an eigenvector $\mu \in V^\vee$ for the transpose $\theta^\vee : V^\vee \to V^\vee$; we let $\lambda \in \mathbf{e}^\times$ denote its eigenvalue. Note that by the non-resonance assumption, $\lambda \notin \mathcal{R}_X$.

We now obtain a continuous homomorphism:

$$\begin{tikzcd} \pi_1^{\text{ét}}(X) \arrow[r, "\rho_0"] \arrow[rrr, bend right, "\overline{\rho}_0"'] & H_\circ(\mathbf{e}) \arrow[r] & V \arrow[r, "\mu"] & \mathbf{e} \end{tikzcd}$$

that by (4.6) satisfies:

$$\overline{\rho}_0({}^F g) = \lambda \cdot \overline{\rho}_0(g). \tag{4.7}$$

We also remark that $\mathrm{Ker}(H_\circ(\mathbf{e}) \to V \to \mathbf{e})$ is the set of $\mathbf{e}$-points of an algebraic subgroup of $H_\circ$, so by definition of the latter, the homomorphism ρ_0 must be non-trivial.

Now $\overline{\rho}_0$ extends to a non-zero $\mathbf{e}$-linear map $H_1^{\text{ét}}(X, \mathbf{e}) \to \mathbf{e}$, i.e., it comes from a non-zero cohomology class $\eta \in H^1_{\text{ét}}(X, \mathbf{e})$. As $g \mapsto {}^F g$ induces the (geometric) Frobenius on $H_1^{\text{ét}}(X, \mathbf{e})$ (see (4.1)), (4.7) means:

$$\phi_X(\eta) = \lambda \cdot \eta.$$

This contradicts the non-resonance assumption, so we conclude that $V = 0$, as was desired.

4.5. Setup for the proof of Theorem 3.5.2.1. We now begin the proof of Theorem 3.5.2.1.

4.5.1. Recall that our objective is to define the commutative diagram (3.11) and verify certain properties of it.

We begin by defining a certain function $\delta : \check{T} \to \mathbf{A}^1$ as:

$$\delta(t) := \prod_{\lambda \in \mathcal{R}_X} \mathrm{ch}_{\check{\mathfrak{g}}}(t, \lambda), \; t \in \check{T}$$

where we use notation as in §4.3.2, and are considering $\check{\mathfrak{g}}$ as a representation of $\check{T}$ via the adjoint action. More explicitly, we have:

$$\delta(t) = \prod_{\lambda \in \mathcal{R}_X} (1 - \lambda)^{\dim(\check{T})} \prod_{\check{\alpha} \in \check{\Delta}} (\check{\alpha}(t) - \lambda)$$

where we consider $\check{\alpha}$ as a map $\check{T} \to \mathbf{G}_m \subseteq \mathbf{A}^1$.

Clearly δ is a W-invariant morphism, so induces a map $\check{T}/\!/W \to \mathbf{A}^1$; we also denote this function by δ.

4.5.2. We now form a commutative diagram:

$$\begin{tikzcd}
 & \mathrm{LS}_{\check{B}}^{\mathrm{arthm},\circ} \arrow[dl] \arrow[d, "\chi_{\check{B}}"] \arrow[dr] & \\
\mathrm{LS}_{\check{T}}^{\mathrm{arthm},\circ} \arrow[d, "\chi_{\check{T}}"] & \check{B}\overset{\mathrm{ad}}{/}\check{B} \arrow[dl] \arrow[dr] & \mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ} \arrow[d, "\chi_{\check{G}}"] \\
\check{T}\overset{\mathrm{ad}}{/}\check{T} \arrow[dr] & & \check{G}\overset{\mathrm{ad}}{/}\check{G} \arrow[dl] \\
 & \check{T}/\!/W \arrow[d, "\delta"] & \\
 & \mathbf{A}^1 &
\end{tikzcd} \tag{4.8}$$

We now define $\delta_{\check{G}}$ so that its restriction to $\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ}$ is given by the unique map $\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ} \to \mathbf{A}^1$ appearing in the diagram (4.8), and so its restriction[45] to $\mathrm{LS}_{\check{G}}^{\mathrm{arthm}} \setminus \mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ}$ is identically 0. We define $\delta_{\check{T}}$ in exactly the same way, replacing $\check{G}$ by $\check{T}$ everywhere in the previous sentence.

Below, we check that the pair of maps $(\delta_{\check{G}}, \delta_{\check{T}})$ satisfy the conclusions of Theorem 3.5.2.1.

4.5.3. First, the commutative diagram (3.11) clearly exists by (4.8).

4.5.4. Second, we need to check (1) from Theorem 3.5.2.1, i.e., that $\delta_{\check{G}}$ takes a non-zero value at the trivial Weil local system. By construction, it is enough to show $\delta(1) \neq 0$. Clearly $\delta(1) = \prod_{\lambda \in \mathcal{R}_X} (1-\lambda)^{\dim(G)}$, and we recall that $1 \notin \mathcal{R}_X$ (see Remark 4.3.1.1).

4.5.5. Next, we observe that the locus where $\delta_{\check{G}}$ is non-zero is exactly $\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ,\text{non-res}}$ (by definition).

Therefore, by Theorem 4.3.3.1, we have:

$$\Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega)[\delta_{\check{G}}^{-1}] = \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ,\text{non-res}}, \omega) \simeq \Gamma((\check{G}\overset{\mathrm{ad}}{/}\check{G})^{\text{non-res}}, \omega) \tag{4.9}$$

which is concentrated in degree zero because it is a localization of $\Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G}, \omega) \simeq \Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G}, \mathcal{O})$ at a function $\check{G}\overset{\mathrm{ad}}{/}\check{G} \to \mathbf{A}^1$, verifying hypothesis (2) from Theorem 3.5.2.1.

4.5.6. It remains to verify the surjectivity (i.e., Theorem 3.5.2.1 (3)). We will do this in the remainder of the section; here we make some preliminary, orienting remarks.

Recall the setting of Lemma 3.7.1.1. We observe that we have two maps:

$$\Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \omega)[\delta_{\check{T}}^{-1}] \xrightarrow{\mathrm{Eis}^{\mathrm{spec}}} \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}}, \omega)[\delta_{\check{G}}^{-1}] = \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ,\text{non-res}}, \omega) \overset{(4.9)}{\simeq}$$
$$\Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G}, \omega)[(\delta_{\check{G}} \circ \tau_{\check{G}})^{-1}] = \Gamma((\check{G}\overset{\mathrm{ad}}{/}\check{G})^{\text{non-res}}, \omega).$$

and:

$$\Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \omega)[\delta_{\check{T}}^{-1}] \overset{\text{Thm. } 4.3.3.1}{=} \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T}, \omega)[(\delta_{\check{T}} \circ \tau_{\check{T}})^{-1}] \xrightarrow{\mathrm{Eis}^{\mathrm{spec,toy}}}$$
$$\Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G}, \omega)[(\delta_{\check{G}} \circ \tau_{\check{G}})^{-1}] = \Gamma((\check{G}\overset{\mathrm{ad}}{/}\check{G})^{\text{non-res}}, \omega).$$

By Corollary 3.7.2.1, we would be done if these two maps coincided.

This expectation is somewhat too naive: we instead show that they coincide *up to invertible L-values*, which will suffice for our purposes.

[45] We remind that $\mathrm{LS}_{\check{G}}^{\mathrm{restr},\circ} \subseteq \mathrm{LS}_{\check{G}}^{\mathrm{restr}}$ is a connected component, so this process of defining the function on $\mathrm{LS}_{\check{G}}^{\mathrm{arthm},\circ} := \mathrm{LS}_{\check{G}}^{\mathrm{arthm}} \times_{\mathrm{LS}_{\check{G}}^{\mathrm{restr}}} \mathrm{LS}_{\check{G}}^{\mathrm{restr},\circ}$ and setting it to be zero elsewhere is legitimate.

4.6. L-values and traces. In §4.5.6, we made an opaque remark about L-values. In this subsection, we will make a precise connection between categorical traces and L-values; this is the main computational input we will need.

In what follows, we let H denote a unipotent algebraic group over $\mathbf{e}$. (In practice, $H = \check{N}$.)

4.6.1. *Classes and traces.* Suppose $\mathcal{C} \in \mathsf{DGCat}_{\mathrm{cont}}$ is a dualizable DG category and $T : \mathcal{C} \to \mathcal{C}$ is an endofunctor.

Let $\mathcal{C}^c \subseteq \mathcal{C}$ denote the subcategory of compact objects and let $\mathcal{C}^{c,T,\mathrm{lax}}$ denote the category of pairs $(\mathcal{F}, \alpha)$ where $\mathcal{F} \in \mathcal{C}^c$ and $\alpha : \mathcal{F} \to T(\mathcal{F})$ is a morphism in $\mathcal{C}$.

Given some $(\mathcal{F}, \alpha)$ as above, there is a canonical point $\mathrm{cl}(\mathcal{F}, \alpha) \in \Omega^\infty \operatorname{tr}_{\mathcal{C}}(T)$. Indeed, this follows from the functoriality of traces as in §3.2.5; equip Vect with the identity self-map, $\mathcal{F}$ as a functor $\mathsf{Vect} \to \mathcal{C}$, and α as a lax intertwining map, so functoriality gives a map $\mathbf{e} = \operatorname{tr}_{\mathsf{Vect}}(\mathrm{id}_{\mathsf{Vect}}) \to \operatorname{tr}_{\mathcal{C}}(T) \in \mathsf{Vect}$, i.e., a point $\mathrm{cl}(\mathcal{F}, \alpha) \in \Omega^\infty \operatorname{tr}_{\mathcal{C}}(T)$.

More generally, we recall that there is a map $\mathrm{cl}(-) : K(\mathcal{C}^{c,T,\mathrm{lax}}) \to \operatorname{tr}_{\mathcal{C}}(T) \in \mathsf{Spectra}$ of spectra from the K-theory spectrum of $\mathcal{C}^{c,T,\mathrm{lax}}$ to the trace of T (with the latter considered as a spectrum via the forgetful functor $\mathsf{Vect} \to \mathsf{Spectra}$).

Notation 4.6.1.1. Note that $\mathsf{Vect}^{c,\mathrm{id},\mathrm{lax}} = \{W \in \mathsf{Vect}^c, \phi : W \to W\}$ is a symmetric monoidal category and as such acts canonically on $\mathcal{C}^{c,T,\mathrm{lax}}$ in the above setting. Explicitly, for $(W, \phi) \in \mathsf{Vect}^{c,\mathrm{id},\mathrm{lax}}$ and $(\mathcal{F}, \alpha) \in \mathcal{C}^{c,T,\mathrm{lax}}$, $W \otimes \mathcal{F}$ is equipped with the endomorphism $\phi \otimes \mathrm{id} + \mathrm{id} \otimes \alpha$.

Under the class map, one has:

$$\mathrm{cl}(W \otimes \mathcal{F}, \phi \otimes \mathrm{id} + \mathrm{id} \otimes \alpha) = \operatorname{tr}_W(\phi) \cdot \mathrm{cl}(\mathcal{F}, \alpha). \tag{4.10}$$

(We do not need this, but this identity can easily be upgraded to a suitable statement at the level of spectra.)

4.6.2. *Statement of the problem.* Recall that H is unipotent. By [**2**] Proposition 3.3.2, $\mathrm{LS}_H^{\mathrm{restr}}$ is a quasi-compact algebraic stack; in particular, its structure sheaf $\mathcal{O}_{\mathrm{LS}_H^{\mathrm{restr}}} \in \mathsf{QCoh}(\mathrm{LS}_H^{\mathrm{restr}})$ is compact (unlike for non-unipotent groups). By abuse of notation, we will let $\mathcal{O}_{\mathrm{LS}_H^{\mathrm{restr}}}$ denote the "same" object of $\mathsf{IndCoh}(\mathrm{LS}_H^{\mathrm{restr}})$ under the fully faithful embedding $\mathsf{QCoh}(\mathrm{LS}_H^{\mathrm{restr}}) \hookrightarrow \mathsf{IndCoh}(\mathrm{LS}_H^{\mathrm{restr}})$ (usually denoted "Ξ" in the literature on IndCoh).

We have a map $\tau_0 : \mathbf{B}H \to \mathrm{LS}_H^{\mathrm{restr}}$ corresponding to the trivial local system. We can then form $\tau_{0,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}H}) \in \mathsf{IndCoh}(\mathrm{LS}_H^{\mathrm{restr}})$.

Note that both objects $\mathcal{O}_{\mathrm{LS}_H^{\mathrm{restr}}}, \tau_{0,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}H})$ are coherent and carry obvious canonical Frobenius equivariant structures. Therefore, we may form their classes:[46]

$$\mathrm{cl}(\mathcal{O}_{\mathrm{LS}_H^{\mathrm{restr}}}, \alpha), \mathrm{cl}(\tau_{0,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}H}), \alpha) \in \Omega^\infty \Gamma(\mathrm{LS}_H^{\mathrm{arthm}}, \omega).$$

Our goal is to compare these two classes.

4.6.3. *An L-value.* Let $\zeta_X(t) = \sum_{n \geqslant 0} |X_0(\mathbf{F}_q^n)| t^n \in \mathbf{Q}(t) \subseteq \mathbf{Q}((t))$ denote the ζ-function of the curve X. We remind that the ζ-function has the form:

$$\zeta_X(t) = \frac{p_X(t)}{(1-t)(1-qt)}, \qquad p_X(t) \in \mathbf{Q}[t].$$

Let $\zeta_X^\star(t) = (1-t) \cdot \zeta_X(t)$. By the Weil conjectures, $\zeta_X^\star(1)$ is non-zero and so equals the leading term of the Taylor expansion[47] of $\zeta_X(t)$ at $t = 1$.

4.6.4. *Main lemma.* We will prove:

Lemma 4.6.4.1. *There exists an equivalence:*

$$\mathrm{cl}(\tau_{0,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}H}), \alpha) \simeq \zeta_X^\star(1)^{\dim H} \cdot \mathrm{cl}(\mathcal{O}_{\mathrm{LS}_H^{\mathrm{restr}}}, \alpha) \in \Omega^\infty \Gamma(\mathrm{LS}_H^{\mathrm{arthm}}, \omega).$$

Less homotopically, this result simply means that the images of the two points above in the set $\pi_0(\Omega^\infty \Gamma(\mathrm{LS}_H^{\mathrm{arthm}}, \omega)) = H^0 \Gamma(\mathrm{LS}_H^{\mathrm{arthm}}, \omega)$ are equal.

[46]By unipotence of H, note that every point of $\mathrm{LS}_H^{\mathrm{arthm}}$ is non-resonant, i.e., the map $H \overset{\mathrm{ad}}{/} H \to \mathrm{LS}_H^{\mathrm{arthm}}$ is an isomorphism.

[47]Note that – unlike in number theory – we are expanding in the variable $t = q^{-s}$ rather than in s itself.

Proof.

Step 1. We begin with a toy model.

Let V and W be finite-dimensional vector spaces equipped with endomorphisms ϕ_V and ϕ_W.

Let $\mathcal{Y}$ denote the stack (over $\mathbf{e}$) $V \times \Omega_0 W$, which we consider equipped with the self-map $\phi = \phi_V \times \Omega_0 \phi_W$; here $\Omega_0 W$ is the derived loop space $0 \times_W 0$.

We let e.g. $\mathcal{O}_0 \in \mathsf{Coh}(\mathcal{Y})$ denote the structure sheaf at the origin, $\mathcal{O}_V \in \mathsf{Coh}(\mathcal{Y})$ denote the structure sheaf of $V \subseteq \mathcal{Y}$, etc.

We use the category $\mathsf{Coh}(\mathcal{Y})^{\phi_*,\mathrm{lax}}$ of lax ϕ_*-equivariant coherent sheaves on $\mathcal{Y}$, i.e., $\mathsf{Coh}(\mathcal{Y})^{\phi_*,\mathrm{lax}} = \{\mathcal{F} \in \mathsf{Coh}(\mathcal{Y}), \alpha : \mathcal{F} \to \phi_*(\mathcal{F})\}$ (see §4.6.1).

Koszul resolutions provide identities:

$$\begin{aligned} \sum (-1)^i [\Lambda^i W^\vee \otimes \mathcal{O}_V] &\simeq [\mathcal{O}_{\mathcal{Y}}] \\ \sum (-1)^j [\Lambda^j V^\vee \otimes \mathcal{O}_V] &\simeq [\mathcal{O}_0]. \end{aligned} \tag{4.11}$$

Here the notation means the following. First, $\Lambda^i W^\vee \otimes \mathcal{O}_{\mathcal{Y}} \in \mathsf{Coh}(\mathcal{Y})^{\phi_*,\mathrm{lax}}$ is equipped with the lax equivariant structure from Notation 4.6.1.1, where $\Lambda^i W^\vee$ is equipped with the endomorphism $\Lambda^i \phi_W^\vee$; similar notation holds for $\Lambda^j V^\vee \otimes \mathcal{O}_V$. The notation $[-]$ is used for the class in the K-theory spectrum[48] $K(\mathsf{Coh}(\mathcal{Y})^{\phi_*,\mathrm{lax}})$.

By (4.10), we find:

$$\begin{gathered} \mathrm{cl}(\mathcal{O}_0, \alpha) = \sum (-1)^j \operatorname{tr}(\Lambda^j \phi_V^\vee) \cdot \mathrm{cl}(\mathcal{O}_V, \alpha) = \sum (-1)^j \operatorname{tr}(\Lambda^j \phi_V) \cdot \mathrm{cl}(\mathcal{O}_V, \alpha) = \\ \det(\mathrm{id}_V - \phi_V) \cdot \mathrm{cl}(\mathcal{O}_V, \alpha) \in \Omega^\infty \Gamma(\mathcal{Y}^\phi, \omega) \end{gathered}$$

for $\mathcal{Y}^\phi$ the derived fixed points.

Similarly, we have:

$$\mathrm{cl}(\mathcal{O}_{\mathcal{Y}}, \alpha) = \det(\mathrm{id}_W - \phi_W) \cdot \mathrm{cl}(\mathcal{O}_V, \alpha) \in \Omega^\infty \Gamma(\mathcal{Y}^\phi, \omega).$$

Comparing these two identities, we obtain:

$$\det(\mathrm{id}_W - \phi_W) \cdot \mathrm{cl}(\mathcal{O}_0, \alpha) = \det(\mathrm{id}_V - \phi_V) \cdot \mathrm{cl}(\mathcal{O}_{\mathcal{Y}}, \alpha).$$

Now assume that $\det(\mathrm{id}_W - \phi_W)$ is non-zero, so is invertible in the field $\mathbf{e}$. We obtain:

$$\mathrm{cl}(\mathcal{O}_0, \alpha) = \frac{\det(\mathrm{id}_V - \phi_V)}{\det(\mathrm{id}_W - \phi_W)} \cdot \mathrm{cl}(\mathcal{O}_{\mathcal{Y}}, \alpha) \in \Omega^\infty \Gamma(\mathcal{Y}^\phi, \omega).$$

Taking $V = H^1_{\text{ét}}(X)$ and $W = H^2_{\text{ét}}(X)$ equipped with their Frobenii endomorphisms, we observe that:

$$\frac{\det(\mathrm{id}_V - \phi_V)}{\det(\mathrm{id}_W - \phi_W)} = \zeta_X^\star(1)$$

by Grothendieck's trace formula.

Step 2. Next, suppose we are in the following more general setup.

We suppose that $\mathcal{Z}$ is a QCA stack equipped with an endomorphism $\phi = \phi_{\mathcal{Z}}$ and is equipped with a quasi-smooth map $\mathcal{Z} \to \mathcal{Y} = V \times \Omega_0 W$ intertwining the maps ϕ.

We let $\mathcal{Z}_0$ denote the fiber of $\mathcal{Z}$ over $0 \in \mathcal{Y}$. We note that $\mathcal{Z}_0$ is eventually coconnective, so $\mathcal{O}_{\mathcal{Z}_0}$ is a coherent sheaf on $\mathcal{Z}$.

The previous analysis then shows:

$$\mathrm{cl}(\mathcal{O}_{\mathcal{Z}_0}, \alpha) = \frac{\det(\mathrm{id}_V - \phi_V)}{\det(\mathrm{id}_W - \phi_W)} \cdot \mathrm{cl}(\mathcal{O}_{\mathcal{Z}}, \alpha) \in \Omega^\infty \Gamma(\mathcal{Z}^\phi, \omega) \tag{4.12}$$

(assuming 1 is not an eigenvalue of ϕ_W).

[48] One could also simply use Grothendieck groups for our purposes.

STEP 3. We now wish to apply the above formalism to deduce our claim.

Choose a nested sequence $\{1\} = H_0 \subseteq H_1 \subseteq \ldots \subseteq H_r = H$ of subgroups with each H_i normal in H and $H_{i+1}/H_i \simeq \mathbf{G}_a$. We remark that $r = \dim(H)$.

Then define algebraic stacks:

$$\mathcal{Z}_i := \mathrm{LS}_H^{\mathrm{restr}} \underset{\mathrm{LS}_{H/H_i}^{\mathrm{restr}}}{\times} \mathbf{B}(H/H_i)$$

where $\mathbf{B}(H/H_i) \to \mathrm{LS}_{H/H_i}^{\mathrm{restr}}$ is the map τ_0, i.e., it corresponds to trivial H/H_i-local systems on X. Observe that $\mathcal{Z}_r = \mathrm{LS}_H^{\mathrm{restr}}$, $\mathcal{Z}_0 = \mathbf{B}H$, and we have closed embeddings:

$$\mathcal{Z}_0 \hookrightarrow \mathcal{Z}_1 \hookrightarrow \ldots \hookrightarrow \mathcal{Z}_r.$$

We can rewrite the definition of $\mathcal{Z}_i$ as follows. Note that H/H_i acts on the classifying stack $\mathbf{B}H_i$; formally, this is encoded by the fiber sequence $\mathbf{B}H_i \to \mathbf{B}H \to \mathbf{B}(H/H_i)$. Unwinding the definitions, this induces an action of H/H_i on $\mathrm{LS}_{H_i}^{\mathrm{restr}}$. We then have:

$$\mathcal{Z}_i \simeq \mathrm{LS}_{H_i}^{\mathrm{restr}}/(H/H_i).$$

Now for each i, we have a diagram:

$$\begin{tikzcd}
H/H_i \arrow[r] \arrow[d, phantom, "\curvearrowright"] & H/H_{i+1} \arrow[r] \arrow[d, phantom, "\curvearrowright"] & \mathbf{BG}_a \arrow[d, phantom, "\curvearrowright"] \\
\mathrm{LS}_{H_i}^{\mathrm{restr}} \arrow[r] & \mathrm{LS}_{H_{i+1}}^{\mathrm{restr}} \arrow[r] & \mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}}
\end{tikzcd}$$

where the rows are fiber sequences and the top row is a fiber sequence of groups. Here the action of $\mathbf{B}G_a$ on $\mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}}$ is induced by the homomorphism of group stacks $\mathbf{BG}_a \to \mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}}$ corresponding to pullback of local systems along $X \to \mathrm{Spec}(k)$ (i.e., the map τ_0 for $\mathbf{G}_a$). Passing to quotients in this diagram and identifying[49] $\mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}} \simeq \mathbf{BG}_a \times H^1_{\text{ét}}(X) \times \Omega_0 H^2_{\text{ét}}(X)$, we obtain a fiber square:

$$\begin{tikzcd}
\mathcal{Z}_i \arrow[r] \arrow[d] & \mathcal{Z}_{i+1} \arrow[d] & \\
\mathrm{Spec}(k) \arrow[r] & \mathrm{LS}_{\mathbf{G}_a}^{\mathrm{restr}}/\mathbf{BG}_a \arrow[r, equal] & H^1_{\text{ét}}(X) \times \Omega_0 H^2_{\text{ét}}(X)
\end{tikzcd}$$

We now obtain the result by induction from the previous step.[50]

□

4.6.5. *An extended digression: divergent series via categorical traces.* We explain a general format for thinking about the above proof of Lemma 4.6.4.1. This material is informal and may be skipped. However, we believe it is an important philosophical point that we wish to highlight.

Roughly speaking, the idea is that so-called *categorical functional analysis* (e.g., fine considerations about distinctions between Perf and Coh) relate to actual analysis (e.g., summing infinite series) via categorical traces. Strikingly, we will see that Hochschild homology allows us to sometimes "correctly" evaluate infinite sums without ever mentioning a topology on the field $\mathbf{e}$ in which they occur.

We consider the following geometric setup. Let $\mathcal{Y}$ be an algebraic stack (over $\mathbf{e}$), which we assume is quasi-smooth and QCA. Assume $\mathcal{Y}$ is equipped with a self-map $\phi : \mathcal{Y} \to \mathcal{Y}$. The functor:

$$\Upsilon_{\mathcal{Y}} : \mathsf{QCoh}(\mathcal{Y}) \to \mathsf{IndCoh}(\mathcal{Y})$$
$$\mathcal{F} \mapsto \mathcal{F} \underset{\mathcal{O}_{\mathcal{Y}}}{\otimes} \omega_{\mathcal{Y}}$$

preserves compact objects and intertwines the self-maps ϕ^* and $\phi^!$ of the source and target. Moreover, this functor is a morphism of $\mathsf{QCoh}(\mathcal{Y})$-module categories.

[49] We note that by purity, there is a *canonical* such splitting compatible with Frobenius.

[50] Formally, the induction should be done on K-theory classes, generalizing (4.11). We map to Hochschild homology only at the end.

Now recall (e.g., [**15**] §3.8.8) that for a dualizable $\mathsf{QCoh}(\mathcal{Y})$-module category $\mathcal{C}$ with an endofunctor $T : \mathcal{C} \to \mathcal{C}$ suitably compatible with ϕ^*, there is a canonical object:

$$\mathrm{tr}^{\mathrm{enh}}(T) \in \mathsf{QCoh}(\mathcal{Y}^\phi)$$

with the basic property that $\Gamma(\mathcal{Y}, \mathrm{tr}^{\mathrm{enh}}(T)) = \mathrm{tr}(T) \in \mathsf{Vect}$. This construction satisfies the usual functoriality properties for traces. We have:[51]

$$\mathrm{tr}^{\mathrm{enh}}(\phi^*) = \mathcal{O}_{\mathcal{Y}^\phi}$$
$$\mathrm{tr}^{\mathrm{enh}}(\phi^!) = \omega_{\mathcal{Y}^\phi}$$

and then $\Upsilon_{\mathcal{Y}}$ yields a canonical map:

$$\tau : \mathcal{O}_{\mathcal{Y}^\phi} = \mathrm{tr}^{\mathrm{enh}}(\phi^*) \to \mathrm{tr}^{\mathrm{enh}}(\phi^!) = \omega_{\mathcal{Y}} \in \mathsf{QCoh}(\mathcal{Y}^\phi).$$

(This map τ can be thought of as a weak Calabi-Yau structure on the derived fixed points.)

We let $\mathcal{Y}^{\phi,\mathrm{good}} \subseteq \mathcal{Y}^\phi$ denote the locus of points where τ is an isomorphism. Note that $\mathcal{Y}^{\phi,\mathrm{good}}$ contains $(\mathcal{Y}^{\mathrm{sm}})^\phi$ (the fixed points of the smooth locus of $\mathcal{Y}$) but in general is larger: one can in fact verify that $\mathcal{Y}^{\phi,\mathrm{good}}$ is exactly the quasi-smooth locus of $\mathcal{Y}^\phi$.

Now, any perfect object $\mathcal{F} \in \mathsf{Perf}(\mathcal{Y})$ with a self map $\alpha : \mathcal{F} \to \phi^*(\mathcal{F})$ yields a class:

$$\mathrm{cl}(\mathcal{F}, \alpha)_{\mathsf{QCoh}(\mathcal{Y})} \in \Omega^\infty \Gamma(\mathcal{Y}^\phi, \mathcal{O})$$

i.e., a function on the fixed points $\mathcal{Y}^\phi$ of ϕ. In this notation, we use the subscript $\mathrm{cl}(-,-)_{\mathsf{QCoh}}$ to emphasize that we are considering $\mathcal{F}$ as an object of QCoh (this will be an important distinction soon). This function can be understood quite explicitly; at a point $y \in \mathcal{Y}^\phi$, we take the trace of the resulting map:

$$\beta_y : y^*(\mathcal{F}) \xrightarrow{\alpha} y^*(\phi^*(\mathcal{F})) = \phi(y)^*(\mathcal{F}) \simeq y^*(\mathcal{F}) \tag{4.13}$$

where the last isomorphism uses the identification $y \simeq \phi(y)$ implicit in y being a fixed point. In other words, we have:

$$\mathrm{cl}(\mathcal{F}, \alpha)_{\mathsf{QCoh}(\mathcal{Y})} = (y \mapsto \mathrm{tr}(\beta_y)).$$

Now suppose instead that $\mathcal{F} \in \mathsf{Coh}(\mathcal{Y})$, though still equipped with a map $\alpha : \mathcal{F} \to \phi^*(\mathcal{F})$. Because $\mathcal{F}$ may not be compact in $\mathsf{QCoh}(\mathcal{Y})$, we cannot form its class in $\Gamma(\mathcal{Y}^\phi, \mathcal{O})$ any longer. However, we can twist and form $\mathcal{F} \otimes \omega_{\mathcal{Y}}$, which lies in Coh because $\mathcal{Y}$ is quasi-smooth (so Gorenstein). We then obtain a map:

$$\widetilde{\alpha} = \alpha \otimes \mathrm{id} : \mathcal{F} \otimes \omega_{\mathcal{Y}} \to \phi^*(\mathcal{F}) \otimes \omega_{\mathcal{Y}} = \phi^!(\mathcal{F} \otimes \omega_{\mathcal{Y}}).$$

Therefore, we can form the class:

$$\mathrm{cl}(\mathcal{F} \otimes \omega_{\mathcal{Y}}, \widetilde{\alpha})_{\mathsf{IndCoh}} \in \Gamma(\mathcal{Y}^\phi, \omega^\phi).$$

Tautologically, in the special case where $\mathcal{F} \in \mathsf{Perf}(\mathcal{Y}) \subseteq \mathsf{Coh}(\mathcal{Y})$, we have:

$$\mathrm{cl}(\mathcal{F} \otimes \omega_{\mathcal{Y}}, \widetilde{\alpha})_{\mathsf{IndCoh}} = \tau(\mathrm{cl}(\mathcal{F}, \alpha)_{\mathsf{QCoh}}). \tag{4.14}$$

Following this equation, we define the *regularized class*:

$$\mathrm{cl}^{\mathrm{reg}}(\mathcal{F}, \alpha)_{\mathsf{QCoh}} \in \Omega^\infty \Gamma(\mathcal{Y}^{\phi,\mathrm{good}}, \mathcal{O})$$

as the image of $\mathrm{cl}(\mathcal{F} \otimes \omega_{\mathcal{Y}}, \widetilde{\alpha})_{\mathsf{IndCoh}}$ under the composition:

$$\Gamma(\mathcal{Y}^\phi, \omega) \to \Gamma(\mathcal{Y}^{\phi,\mathrm{good}}, \omega) \overset{\tau^{-1}}{\simeq} \Gamma(\mathcal{Y}^{\phi,\mathrm{good}}, \mathcal{O}).$$

By construction, this regularized class coincides with (the restriction to $\mathcal{Y}^{\phi,\mathrm{good}}$ of) $\mathrm{cl}(\mathcal{F}, \alpha)_{\mathsf{QCoh}}$ when $\mathcal{F}$ is perfect.

Suppose $y \in \mathcal{Y}^\phi$. The map (4.13) still makes sense. However, if $\mathcal{F}$ is not perfect near y, then while the complex (4.13) is finite-dimensional in each degree, it is unbounded from below, so the trace of β_y is not well defined. We define the *regularized trace*:

$$\mathrm{tr}^{\mathrm{reg}}(\beta_y) := (\mathrm{cl}^{\mathrm{reg}}(\mathcal{F}, \alpha)_{\mathsf{QCoh}})(y)$$

---

[51] In this formula, we consider $\omega_{\mathcal{Y}^\phi}$ as an object of QCoh, not of IndCoh. In other words, we implicitly are taking the "true" dualizing sheaf in IndCoh and applying the forgetful functor $\Psi : \mathsf{IndCoh}(\mathcal{Y}^\phi) \to \mathsf{QCoh}(\mathcal{Y}^\phi)$ to it. As $\omega_{\mathcal{Y}^\phi}$ is a line bundle, this is a quite mild thing to have done, so we do not specifically demarcate it in the notation.

as the value of the regularized class at $y \in \mathcal{Y}^\phi$.

Heuristically, the regularized trace can be thought of as assigning an actual value to the infinite sum:

$$\sum_{i\in\mathbf{Z}} (-1)^i \operatorname{tr}(H^i(\beta_y)) : H^i(y^*(\mathcal{F})) \to H^i(y^*(\mathcal{F})) \in \mathbf{e} \tag{4.15}$$

where we reiterate that the summands are each well-defined, the summands vanish for $i \gg 0$, but generally, an infinite number of summands appear.

Example 4.6.5.1. Let us explain how this works in the simplest possible case. Suppose $\mathcal{Y} = \Omega_0\mathbf{A}^1$ and ϕ is multiplication by a number $\lambda \in \mathbf{e}$. Take the sheaf $\mathcal{F}$ to be $\mathcal{O}_0$, the structure sheaf of the point $0 \in \Omega_0\mathbf{A}^1$. Note that 0 is canonically a fixed point of ϕ, so we can think of 0 as a point of $\mathcal{Y}^\phi$. By a standard calculation, $0^*(\mathcal{O}_0)$ has 1-dimensional cohomology in even non-positive cohomological degrees and vanishing cohomology outside these degrees; moreover, the map β_0 acts on $H^{-2i}(0^*(\mathcal{O}_0))$ as multiplication by λ^i. Therefore, the sum from (4.15) is the geometric series $\sum_{i\geqslant 0}\lambda^i$. We emphasize that this is a formal expression; at the moment, λ is an arbitrary element of the field λ and is in no sense "small."

Now suppose $\lambda \neq 1$. Then $0 \in \mathcal{Y}^{\phi,\text{good}}$ (in fact, $\mathcal{Y}^{\phi,\text{good}} = \mathcal{Y}^\phi = \operatorname{Spec}(\mathbf{e}) = \{0\}$). Then the regularized trace $\operatorname{tr}^{\text{reg}}(\beta_0)$ is well-defined, and the (completely elementary) argument from Step 1 from the proof of Lemma 4.6.4.1 calculates:

$$\operatorname{tr}^{\text{reg}}(\beta_0) = \frac{1}{1-\lambda}.$$

In other words, we have given direct, purely algebraic meaning to the geometric series formula $\sum \lambda^i \text{``=''} \frac{1}{1-\lambda}$, which usually requires us to know $\lambda^i \overset{i\to\infty}{\to} 0$ in some suitably analytic sense.

Remark 4.6.5.2 (Regularized traces and functional equations). Suppose now that $\phi : \mathcal{Y} \to \mathcal{Y}$ is in fact an isomorphism. Then $\phi^* = \phi^!$ (say, as functors restricted to Perf or Coh). Therefore, for $\mathcal{F} \in \mathsf{Perf}(\mathcal{Y})$ with $\alpha : \mathcal{F} \to \phi^*(\mathcal{F})$, we also obtain a mp $\alpha' : \mathcal{F} \to \phi^!(\mathcal{F})$. For $y \in \mathcal{Y}^\phi$, we obtain a canonical map:

$$\gamma_y : y^!(\mathcal{F}) \to y^!(\mathcal{F})$$

defined in the same way as β_y. As $\mathcal{F}$ is perfect, we have:

$$y^!(\mathcal{F}) = y^!(\mathcal{F} \otimes \omega_{\mathcal{Y}} \otimes \omega_{\mathcal{Y}}^{-1}) = y^!(\mathcal{F} \otimes \omega_{\mathcal{Y}}) \otimes y^*(\omega_{\mathcal{Y}}^{-1}) = y^*(\mathcal{F}) \otimes y^*(\omega_{\mathcal{Y}})^{\otimes -1}.$$

This map intertwines γ_y (for $\mathcal{F}$) with β_y (for both $\mathcal{F}$ and $\omega_{\mathcal{Y}}$). If we set ϵ_y to be the trace of the map:

$$y^*(\omega_{\mathcal{Y}}) \to y^*(\omega_{\mathcal{Y}})$$

constructed using γ_y (and the obvious isomorphism $\omega_{\mathcal{Y}} \simeq \phi^!(\omega_{\mathcal{Y}}) = \phi^*(\omega_{\mathcal{Y}})$), we find:

$$\epsilon_y \cdot \operatorname{tr}(\gamma_y) = \operatorname{tr}(\beta_y).$$

Now we can define $\operatorname{tr}^{\text{reg}}(\gamma_y)$ exactly as we did for coherent $\mathcal{F}$ when $y \in \mathcal{Y}^{\phi,\text{good}}$. We obtain a tautological "functional equation:"

$$\epsilon_y \cdot \operatorname{tr}^{\text{reg}}(\gamma_y) = \operatorname{tr}^{\text{reg}}(\beta_y)$$

(where ϵ_y is thought of as an ϵ-factor).

Let us see how this logic plays out in the setting of Example 4.6.5.1. We should have $\lambda \neq 0$ so that ϕ is an isomorphism. We note that $0^!(\mathcal{O}_0)$ has cohomology in even non-negative degrees, and the action of γ_0 on $H^{2i}(0^!(\mathcal{O}_0))$ is multiplication by λ^{-i}. Also, $0^*(\omega) = \mathbf{e}[-1]$ with "γ" operator multiplication by λ^{-1}. Therefore, the regularized trace $\operatorname{tr}^{\text{reg}}(\gamma_0)$ heuristically makes sense of the sum:

$$\operatorname{tr}^{\text{reg}}(\gamma_0) \text{``=''} \sum_{i\geqslant 0} \lambda^{-i}$$

while $\epsilon_0 = -\lambda^{-1}$.

Therefore, in this case, our functional equation heuristically yields:

$$-\lambda^{-1} \sum_{i\geqslant 0} \lambda^{-i} \text{``=''} \varepsilon_0 \operatorname{tr}^{\text{reg}}(\gamma_0) = \operatorname{tr}^{\text{reg}}(\beta_0) \text{``=''} \sum_{i\geqslant 0} \lambda^i.$$

This resulting equation $-\lambda^{-1}\sum_{i\geqslant 0}\lambda^{-i}$ "=" $\sum_{i\geqslant 0}\lambda^i$ is a favorite from the world of divergent series; over $\mathbf{C}$, the left hand side is defined for $|\lambda|>1$ while the right hand side is defined for $|\lambda|<1$, but, of course, the analytic continuations of these two functions coincide on their domains.

We note that this sort of manipulation with divergent series is closely related to the functional equation for the ζ-function of an algebraic curve.

Remark 4.6.5.3. It would be of great interest to interpret categorically some analytic aspects of the analytic theory of automorphic forms over function fields using some version of the above ideas.

4.6.6. *Variant.* In practice, we need a slight extension of the discussion of §4.6.4.

Let H be a unipotent group as before, and now let S be a torus acting on H by automorphisms. (In practice, $S=\check{T}$ acting on $H=\check{N}$.) For brevity, we let Q denote the semi-direct product $S\ltimes H$.

In this case, we define a rational map:

$$\zeta^\star_{X,H,S}: S \dashrightarrow \mathbf{A}^1$$

via the formula:

$$(s\in S)\mapsto \frac{\det\left(\mathrm{id}-\mathrm{Ad}_{s^{-1}}\otimes\phi_X \curvearrowright \mathfrak{h}\otimes H^1_{\text{ét}}(X)\right)}{\det\left(\mathrm{id}-\mathrm{Ad}_{s^{-1}}\otimes\phi_X \curvearrowright \mathfrak{h}\otimes H^2_{\text{ét}}(X)\right)}.$$

Here ϕ_X is the Frobenius acting on étale cohomology of X while we abuse notation somewhat in letting Ad_{-} denote the action of S on $\mathfrak{h}$ coming from the action of S on H.

Remark 4.6.6.1. Note that $\zeta^\star_{X,H,S}$ is defined at $1\in S$ and takes the value $\zeta^\star_X(1)^{\dim H}$ there (see Lemma 4.6.4.1).

Remark 4.6.6.2. Suppose $\mu_1,\ldots,\mu_r: S\to\mathbf{G}_m$ are the characters of S appearing in its representation $\mathfrak{h}$, counted with multiplicities (so $r=\dim(H)$). Then we have:

$$\zeta^\star_{X,H,S}(s)=\prod_{i=1}^r \zeta^\star_X(\mu_i(s^{-1})).$$

In particular, the domain of definition of $\zeta^\star_{X,H,S}$ is $\cap_i\{s\in S \mid \mu_i(s)\neq q\}$, and $\zeta^\star_{X,H,S}$ is (defined and) invertible on $\cap_i\{s\in S\mid \mu_i(s)\notin\mathcal{R}_X\}$.

It will be convenient also to introduce the notation:

$$p_{X,H,S}(s)=\det\left(\mathrm{id}-\mathrm{Ad}_{s^{-1}}\otimes\phi_X\curvearrowright\mathfrak{h}\otimes H^1_{\text{ét}}(X)\right)=\prod_{i=1}^r p_X(\mu_i(s^{-1}))$$
$$q_{X,H,S}(s)=\det\left(\mathrm{id}-\mathrm{Ad}_{s^{-1}}\otimes\phi_X\curvearrowright\mathfrak{h}\otimes H^2_{\text{ét}}(X)\right)=\prod_{i=1}^r (1-q\cdot\mu_i(s^{-1}))$$

so $p_{X,H,S}$ and $q_{X,H,S}$ are (regular) functions on S with $\zeta^\star_{X,H,S}=\frac{p_{X,H,S}}{q_{X,H,S}}$.

We introduce the notation:

$$\mathrm{LS}^{\mathrm{restr}}_{Q;S}:=\mathrm{LS}^{\mathrm{restr}}_Q\underset{\mathrm{LS}^{\mathrm{restr}}_S}{\times}\mathbf{B}S=(\mathrm{LS}^{\mathrm{restr}}_H)/S.$$

We observe that $\mathrm{LS}^{\mathrm{restr}}_{Q;S}$ is a quasi-compact algebraic stack (by unipotence of H). We let $\mathrm{LS}^{\mathrm{arthm}}_{Q;S}$ denote the Frobenius fixed points of $\mathrm{LS}^{\mathrm{restr}}_{Q;S}$. Explicitly, we have:

$$\mathrm{LS}^{\mathrm{arthm}}_{Q;S}=\mathrm{LS}^{\mathrm{arthm}}_Q\underset{\mathrm{LS}^{\mathrm{arthm}}_S}{\times} S\overset{\mathrm{ad}}{/}S.$$

We let $\tau_1:\mathbf{B}Q\to\mathrm{LS}^{\mathrm{restr}}_{Q;S}$ denote the evident map.

Remark 4.6.6.3. To be more explicit, we remind that $S\overset{\mathrm{ad}}{/}S\xrightarrow{\simeq}\mathrm{LS}^{\mathrm{arthm},\circ}_S$ by Theorem 4.3.3.1, recalling that S is a torus. Therefore, $\mathrm{LS}^{\mathrm{arthm}}_{Q;S}$ is the connected component of the identity in $\mathrm{LS}^{\mathrm{arthm}}_Q$.

Lemma 4.6.6.4. *There exists an equivalence:*

$$q_{X,H,S} \cdot \mathrm{cl}(\tau_{1,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}Q}), \alpha) \simeq p_{X,H,S} \cdot \mathrm{cl}(\mathcal{O}_{\mathrm{LS}_{Q;S}^{\mathrm{restr}}}, \alpha) \in \Omega^{\infty}\Gamma(\mathrm{LS}_{Q;S}^{\mathrm{arthm}}, \omega).$$

Remark 4.6.6.5. Informally, the lemma should be understood as saying:

$$\mathrm{cl}(\tau_{1,*}^{\mathsf{IndCoh}}(\mathcal{O}_{\mathbf{B}Q}), \alpha) \simeq \zeta_{X,H,S}^{\star} \cdot \mathrm{cl}(\mathcal{O}_{\mathrm{LS}_{Q;S}^{\mathrm{restr}}}, \alpha)$$

Proof of Lemma 4.6.6.4. The proof is essentially identical to that of Lemma 4.6.4.1. The differences are as follows.

First, in Step 1, one should assume V and W are S-representations, and one should account for the S-action in (4.11). That the dual representations $V^{\vee}$ and $W^{\vee}$ appear in (4.11) accounts for the appearance of s^{-1} rather than s in the definition of $\zeta_{X,H,S}^{\star}$ above.

Second, one should note that the subgroups H_i from the proof of Lemma 4.6.4.1 can be taken to be invariant under the S-action (proof: diagonalize the S-action on $\mathfrak{h}/[\mathfrak{h},\mathfrak{h}]$ and proceed by induction).

Otherwise, the argument proceeds verbatim.

□

4.7. Conclusion. We now return to the setting of §4.5.

4.7.1. Let $\mathrm{LS}_{\check{T},\delta_{\check{T}}\neq 0}^{\mathrm{arthm}} \subseteq \mathrm{LS}_{\check{T}}^{\mathrm{arthm}}$ denote the non-vanishing locus of $\delta_{\check{T}}$.

We have a rational function:

$$\mathrm{LS}_{\check{T}}^{\mathrm{arthm},\circ} \overset{\mathrm{Thm.}\ 4.3.3.1}{=} \check{T} \overset{\mathrm{ad}}{/} \check{T} = \check{T} \times \mathbf{B}\check{T} \xrightarrow{p_1} \check{T} \overset{\zeta_{X,\check{N},\check{T}}}{\dashrightarrow} \mathbf{A}^1$$

that is clearly defined and invertible on $\mathrm{LS}_{\check{T},\delta_{\check{T}}\neq 0}^{\mathrm{arthm}}$. By abuse of notation, we also let $\zeta_{X,\check{N},\check{T}}$ denote the resulting map:

$$\zeta_{X,\check{N},\check{T}} : \mathrm{LS}_{\check{T},\delta_{\check{T}}\neq 0}^{\mathrm{arthm}} \to \mathbf{A}^1 \backslash 0.$$

4.7.2. We now prove the following result:

Theorem 4.7.2.1. *There is a commutative diagram:*

$$\begin{tikzcd}
\Gamma(\mathrm{LS}_{\check{T},\delta_{\check{T}}\neq 0}^{\mathrm{arthm}}, \omega) \arrow[r, equal] & \Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \omega)[\delta_{\check{T}}^{-1}] \arrow[r, "\mathrm{Eis}^{\mathrm{spec}}"] & \Gamma((\check{G} \overset{\mathrm{ad}}{/} \check{G})^{\text{non-res}}, \omega) \arrow[dr, "\mathrm{id}"] & \\
\Gamma(\mathrm{LS}_{\check{T},\delta_{\check{T}}\neq 0}^{\mathrm{arthm}}, \omega) \arrow[u, "\simeq"', "\zeta_{X,\check{N},\check{T}} \cdot -"] \arrow[r, equal] & \Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}, \omega)[\delta_{\check{T}}^{-1}] \arrow[r, "\mathrm{Eis}^{\mathrm{spec,toy}}"] & \Gamma(\check{G} \overset{\mathrm{ad}}{/} \check{G}, \omega)[(\delta_{\check{G}} \circ \tau_{\check{G}})^{-1}] \arrow[r, equal] & \Gamma((\check{G} \overset{\mathrm{ad}}{/} \check{G})^{\text{non-res}}, \omega).
\end{tikzcd}$$

Using Corollary 3.7.2.1, this clearly yields the desired surjectivity from §4.5.6. Therefore, it remains to prove this theorem.

Proof of Theorem 4.7.2.1.

Step 1. The commutative diagram:

$$\begin{tikzcd}
\mathbf{B}\check{B} \arrow[r, "\tau_{\check{B}}"] \arrow[d, "\mathfrak{p}^{\mathrm{toy}}"] & \mathrm{LS}_{\check{B}}^{\mathrm{restr}} \arrow[d, "\mathfrak{p}"] \\
\mathbf{B}\check{G} \arrow[r, "\tau_{\check{G}}"] & \mathrm{LS}_{\check{G}}^{\mathrm{restr}}
\end{tikzcd}$$

of stacks under proper morphisms yields an identification of the resulting two functors:

$$\mathsf{Rep}(\check{B}) \to \mathsf{IndCoh}(\mathrm{LS}_{\check{G}}^{\mathrm{restr}})$$

preserving compact objects.

Passing to traces of Frobenius, this yields a commutative diagram:

$$\begin{tikzcd} \Gamma(\check{B}\overset{\mathrm{ad}}{/}\check{B},\omega) \arrow[r] \arrow[d] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm}},\omega) \arrow[d] \\ \Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G},\omega) \arrow[r] & \Gamma(\mathrm{LS}_{\check{G}}^{\mathrm{arthm}},\omega). \end{tikzcd} \tag{4.16}$$

Step 2. Next, form the commutative square:

$$\begin{tikzcd} \mathbf{B}\check{B} \arrow[r, "\tau_{\check{B}}"] \arrow[d, "\mathfrak{q}^{\mathrm{toy}}"] & \mathrm{LS}_{\check{B}}^{\mathrm{restr}} \arrow[d, "\mathfrak{q}"] \\ \mathbf{B}\check{T} \arrow[r, "\tau_{\check{T}}"] & \mathrm{LS}_{\check{T}}^{\mathrm{restr}} \end{tikzcd}$$

We obtain a natural transformation:

$$\mathfrak{q}^{*,\mathsf{IndCoh}}\tau_{\check{T},*}^{\mathsf{IndCoh}} \to \tau_{\check{B},*}^{\mathsf{IndCoh}}\mathfrak{q}^{\mathrm{toy},*,\mathsf{IndCoh}}$$

of functors:

$$\mathsf{Rep}(\check{T}) \to \mathsf{IndCoh}(\mathrm{LS}_{\check{B}}^{\mathrm{restr}}).$$

But this functor is *not* an isomorphism, so mere functoriality of traces has little to say about it.

Still, we claim that we have a commutative diagram:

$$\begin{tikzcd} \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "p_{X,\check{N},\check{T}}\cdot -"] \arrow[d, equal] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "\mathrm{tr}(\mathfrak{q}^{*,\mathsf{IndCoh}}\tau_{\check{T},*}^{\mathsf{IndCoh}})"] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm}},\omega) \\ \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "q_{X,\check{N},\check{T}}\cdot -"] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega). \arrow[ur, "\mathrm{tr}(\tau_{\check{B},*}^{\mathsf{IndCoh}}\mathfrak{q}^{\mathrm{toy},*,\mathsf{IndCoh}})"'] & \end{tikzcd} \tag{4.17}$$

To construct this diagram, note that the maps are naturally morphisms of $\Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm}},\mathcal{O})$-modules. This algebra clearly acts on $\Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega)$ (the source of the diagram we wish to construct) through its factor:

$$\Gamma(\mathrm{LS}_{\check{T}}^{\mathrm{arthm},\circ},\mathcal{O}) = \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\mathcal{O}).$$

Therefore, it suffices to produce a commutative diagram:

$$\begin{tikzcd} \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "p_{X,\check{N},\check{T}}\cdot -"] \arrow[d, equal] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "\mathrm{tr}(\mathfrak{q}^{*,\mathsf{IndCoh}}\tau_{\check{T},*}^{\mathsf{IndCoh}})"] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm}},\omega) \arrow[r, "\mathrm{proj}"] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm},\circ},\omega) \arrow[d, equal] \\ \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "q_{X,\check{N},\check{T}}\cdot -"] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "\mathrm{tr}(\tau_{\check{B},*}^{\mathsf{IndCoh}}\mathfrak{q}^{\mathrm{toy},*,\mathsf{IndCoh}})"] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm}},\omega) \arrow[r, "\mathrm{proj}"] & \Gamma(\mathrm{LS}_{\check{B}}^{\mathrm{arthm},\circ},\omega) \end{tikzcd}$$

of $\Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\mathcal{O})$-modules (as $\mathrm{LS}_{\check{B}}^{\mathrm{arthm},\circ} = \mathrm{LS}_{\check{B}}^{\mathrm{arthm}} \times_{\mathrm{LS}_{\check{T}}^{\mathrm{arthm}}} \mathrm{LS}_{\check{T}}^{\mathrm{arthm},\circ}$).

Recall that the dualizing sheaf on $H\overset{\mathrm{ad}}{/}H$ is canonically trivial. This trivialization can be constructed as follows: the equivalence $\mathsf{QCoh}(\mathbf{B}H) \xrightarrow{\Xi} \mathsf{IndCoh}(\mathbf{B}H)$ gives an isomorphism on Hochschild homology $\Gamma(H\overset{\mathrm{ad}}{/}H,\mathcal{O}) \xrightarrow{\simeq} \Gamma(H\overset{\mathrm{ad}}{/}H,\omega)$. Let $\mathrm{vol}_H \in \Omega^\infty\Gamma(H\overset{\mathrm{ad}}{/}H,\omega)$ denote the resulting generator – explicitly, it is the class of the identity object of $\mathsf{Rep}(H)$. Therefore, to produce the above diagram, it suffices to provide an isomorphism:[52]

$$p_{X,\check{N},\check{T}} \cdot \mathrm{proj}\left(\mathrm{tr}(\mathfrak{q}^{*,\mathsf{IndCoh}}\tau^{\mathsf{IndCoh}}_{\check{T},*})(\mathrm{vol}_{\check{T}})\right) =$$
$$q_{X,\check{N},\check{T}} \cdot \mathrm{proj}\left(\mathrm{tr}(\tau^{\mathsf{IndCoh}}_{\check{B},*}\mathfrak{q}^{\mathrm{toy},*,\mathsf{IndCoh}})(\mathrm{vol}_{\check{T}})\right) \in \Omega^{\infty}\Gamma(\mathrm{LS}^{\mathrm{arthm},\circ}_{\check{B}},\omega).$$

By construction, we have:

$$\mathrm{tr}(\tau^{\mathsf{IndCoh}}_{\check{B},*}\mathfrak{q}^{\mathrm{toy},*,\mathsf{IndCoh}})(\mathrm{vol}_{\check{T}}) = \mathrm{cl}(\tau^{\mathsf{IndCoh}}_{\check{B},*}\mathcal{O}_{\mathbf{B}\check{B}},\alpha) \in \Omega^{\infty}\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{B}},\omega).$$

Similarly, by base-change, we have:

$$\mathrm{tr}(\mathfrak{q}^{*,\mathsf{IndCoh}}\tau^{\mathsf{IndCoh}}_{\check{T},*})(\mathrm{vol}_{\check{T}}) = \mathrm{cl}(\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{B}}\times_{\mathrm{LS}^{\mathrm{restr}}_{\check{T}}}\mathbf{B}\check{T}},\alpha) = \mathrm{cl}(\mathcal{O}_{\mathrm{LS}^{\mathrm{restr}}_{\check{B};\check{T}}},\alpha) \in \Omega^{\infty}\Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{B}},\omega).$$

So the identity follows from Lemma 4.6.6.4, reminding that $\mathrm{LS}^{\mathrm{arthm}}_{\check{B};\check{T}} = \mathrm{LS}^{\mathrm{arthm},\circ}_{\check{B}}$ (see Remark 4.6.6.3).

Step 3. Concatenating diagrams (4.16) and (4.17), we obtain a commutative diagram:

$$\begin{tikzcd}
\Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "p_{X,\check{N},\check{T}}\cdot -"] \arrow[d, equal] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "\mathrm{tr}(\tau^{\mathsf{IndCoh}}_{\check{T},*})"] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{T}},\omega) \arrow[r, "\mathrm{Eis}^{\mathrm{spec}}"] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}},\omega) \arrow[d, equal] \\
\Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "q_{X,\check{N},\check{T}}\cdot -"] & \Gamma(\check{T}\overset{\mathrm{ad}}{/}\check{T},\omega) \arrow[r, "\mathrm{Eis}^{\mathrm{spec,toy}}"] & \Gamma(\check{G}\overset{\mathrm{ad}}{/}\check{G},\omega). \arrow[r, "\mathrm{tr}(\tau^{\mathsf{IndCoh}}_{\check{G},*})"] & \Gamma(\mathrm{LS}^{\mathrm{arthm}}_{\check{G}},\omega).
\end{tikzcd}$$

This diagram refines the theorem we were supposed to prove.

□

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